

ON THE GROWTH AND STABILITY OF REAL-ANALYTIC FUNCTIONS

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Abstract. This paper addresses the issue of whether integrals of real-analytic functions remain finite under small deformations. An approach based on uniform estimates for certain classes of one-dimensional integrals is introduced. It is powerful enough to recover the stability properties of real integrals in two dimensions which follow from the work of Karpushkin, as well as produce new results in higher dimensions. In dimension three, the new stability results are sharp, as shown by the well-known example of Varchenko.

I. Introduction. There has been considerable interest in recent years in developing a method of stationary phase for degenerate phases, as such a method would be instrumental to a new generation of problems in analysis, probability, and geometry. The earliest work in this direction goes back to Varchenko [23], and some of the more recent progress is in [2, 4, 8, 9, 15] for scalar oscillatory integrals, and in [3, 5, 15, 16, 18] for oscillatory integral operators. The degenerate phase case gives rise, however, to a complex issue which was absent in the traditional nondegenerate case, namely the issue of *stability*. More specifically, consider an estimate of the form

$$(1.1) \quad \left| \int_{\mathbf{R}^n} e^{i\lambda f(x)} \chi(x) dx \right| \leq C_{f,\chi}^{\gamma,\delta} |\lambda|^{-\delta} \log^\gamma(1 + |\lambda|)$$

where χ is a smooth function with compact support, and $C_{f,\chi}^{\gamma,\delta}$ is a constant independent of λ . Stability is the issue of *uniformity* of the estimate (1.1) with respect to parameters, primarily with respect to small deformations of the phase $f(x)$, but also, as in the case of oscillatory integral operators, with respect to certain classes of cut-off functions χ (see the “curved-box lemma” of [16]). A closely related problem is the problem of stability for the integrals

$$(1.2) \quad \int_{\mathbf{B}^{(n)}} |f(x)|^{-\delta} dx$$

where $\mathbf{B}^{(n)} \subseteq \mathbf{R}^n$ is a small ball centered at the origin. In the holomorphic context, this problem is actually central to the solvability of certain complex

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Monge-Ampere equations [19, 20, 22]. The stability of both estimates (1.1) and (1.2) is still rather poorly understood, and we refer to Sections IV and V below for a discussion of what is known so far.

The main goal of this paper is to introduce an approach to this circle of problems based on some *uniform, one-dimensional* estimates, and to apply these estimates to stability in higher dimensions. In particular, we shall show that for $n = 2$ the integral defined by (1.2) is stable with respect to small deformations of f . For $n = 3$, we show that the integral is stable provided $\delta < 2/N$, where N denotes the multiplicity of f at the origin. Varchenko's example (see Section IV) shows that our $n = 3$ result is sharp. The $n = 2$ result can also be derived from an analogous theorem of Karpushkin [8, 9] on the stability of the two-dimensional oscillatory integrals (1.1). Karpushkin's argument is a difficult blend of analysis with the versal theory of deformations, so our approach can be viewed as an elementary alternative.

The basic idea is as follows. Integrals of the form (1.2) can be viewed as a measure of how close the branches of the null variety of $f(x)$ come to one another. As the function f is perturbed, however slightly, the behavior of the individual branches can undergo abrupt transitions. However, the integrals themselves are symmetric functions of all these branches, and, under suitable circumstances, may be expressible in terms of simpler characteristics of f . If $f(x)$ is for example a polynomial $P(x', y)$ in one of the variables, say $x_n = y$, (as is often the case, up to a nonvanishing prefactor, in view of the Weierstrass preparation theorem), then the coefficients of $P(x', y)$ are characteristics of $f(x)$ which are much better behaved under perturbations than its null variety. The above idea can be implemented explicitly in some important ranges for δ , as we shall show in the paper.

Thus we consider monic polynomials $P(y)$ of degree N and of the form

$$(1.3) \quad P(y) = \sum_{i=0}^N a_i y^{N-i} = \prod_{i=1}^N (y - y_i),$$

where $y_i \in \mathbf{C}$, $1 \leq i \leq N$. A first simple uniform estimate is the following estimate on the growth of the distribution function of $P(y)$:

THEOREM 1. *Let $P(y)$ be a monic polynomial of the form (1.3). Then we have*

$$(1.4) \quad |\{y \in \mathbf{R}; |P(y)| \leq \epsilon\}| \leq C_N \max_j \min_{S \ni j} \left(\frac{\epsilon}{\prod_{k \notin S} |y_j - y_k|} \right)^{1/|S|}$$

where S ranges over all subsets of $\{1, 2, \dots, N\}$ which include j , $|S|$ denotes the number of elements in S , and C_N is an absolute constant depending only on the order N of the polynomial.

Our next theorem concerns the integral (1.2) in the case where $f = P$, a monic polynomial of one variable. In general, it is not possible to evaluate the integral in

closed form: We shall show, however, that for certain key ranges of δ , the *size* of the integral has a closed form expression. We first explain the result in a special case which is notationally simple: Fix an integer $N > 0$ and a rational number δ satisfying $0 < \delta < 2/N$ and $\delta^{-1} \notin \mathbf{Z}$. Then there exists a real number $C > 1$ and $\delta' > 0$, depending only on N and δ , and a polynomial $\Phi \in \mathbf{Z}[a_1, \dots, a_N]$, depending only on N and δ , such that

$$\frac{C^{-1}}{|\Phi(a_1, \dots, a_N)|^{\delta'}} \leq \int_{-1}^1 \frac{1}{|P(y)|^\delta} dy \leq \frac{C}{|\Phi(a_1, \dots, a_N)|^{\delta'}}$$

for all monic $P(y) = \sum_{i=0}^N a_i y^{N-i} \in \mathbf{R}[y]$ satisfying $|a_i| \leq 1$ for all i .

To state the estimates in a more general setting, we require the following notation. For each subset S of $\{1, \dots, N\}$, set

$$(1.5) \quad L(S) = \sup_{i,j \in S} \{ |Re(y_i - y_j)|, |Im(y_i)| \}.$$

We define for each integer $k, 0 \leq k < N$, the *kth cluster scale* L_k by

$$(1.6) \quad L_k = \inf_{S, |S|=N-k} L(S).$$

and for each integer $r, 1 \leq r \leq N/2$, the *rth positive discriminant* Δ_k by

$$(1.7) \quad \Delta_r(y_1, \dots, y_N) = \Delta_r = \sup \prod_{\nu=1}^r |y_{i_\nu} - y_{j_\nu}|$$

where the supremum is taken over all $2r$ - *tuples* $(i_1, \dots, i_r, j_1, \dots, j_r) \in \{1, 2, \dots, N\}^{2r}$ with *distinct* entries (i.e., the set $\{i_1, \dots, i_r, j_1, \dots, j_r\}$ has $2r$ elements). We set $\Delta_0 = 1$.

We shall also say that an interval $I \subseteq \mathbf{R}$ is *P-admissible* if the following two conditions are satisfied:

- (i) $Re(y_i) \in I$ for all i ;
- (ii) $|I| \geq L_0$, where $|I|$ denotes the length of I .

For $\delta > 0$, we define

$$(1.8) \quad Q_I(P) = Q_I = \int_I |P(y)|^{-\delta} dy.$$

Henceforth all intervals I are assumed to be *P-admissible* unless indicated explicitly otherwise. In general, we consider N and δ as fixed. Two positive quantities A and B are viewed as equivalent in size, denoted $A \sim B$, if there exists a positive constant $c_{N,\delta}$ depending only on N and δ such that

$$c_{N,\delta}^{-1} A \leq B \leq c_{N,\delta} A.$$

THEOREM 2. (a) For $\delta^{-1} \notin \mathbf{N}$, and in the range $0 < \delta < \frac{2}{N}$ for N even, $0 < \delta < \frac{2}{N-1}$ for N odd, the size of the integral Q_I over P -admissible intervals can be characterized as follows. For $0 < \delta < \frac{1}{N}$, Q_I is of size

$$(1.9) \quad Q_I \sim |I|^{1-N\delta},$$

while for $\frac{1}{N-k} < \delta < \frac{1}{N-k-1}$, $0 \leq k < \frac{N}{2}$, it is of size

$$(1.10) \quad Q_I \sim \frac{1}{(L_0 L_1 \cdots L_{k-1})^\delta L_k^{(N-k)\delta-1}},$$

where the L_k denote the cluster scales of (1.6).

(b) Assume that P is real valued. For each r , $0 \leq r \leq \frac{N}{2}$, let

$$M = \frac{N!}{(N-2r)!}, \quad e = e(r) = 2M!$$

Then there exists a polynomial $D_r(A_1, \dots, A_N) \in \mathbf{Z}[A_1, \dots, A_N]$, such that

$$\Delta_r \sim |D_r(a_1, \dots, a_N)|^{1/e},$$

where the $a_j \in \mathbf{R}$ are the coefficients of the polynomial $P(y)$, i.e., $P(y) = \sum_{i=0}^N a_{N-i}y^i$, $a_0 = 1$.

(c) When $P(y)$ is real-valued, and $\frac{1}{N-k} < \delta < \frac{1}{N-k-1}$, $0 \leq k \leq \frac{N}{2} - 1$, the estimates (1.10) can be rewritten as

$$(1.11) \quad Q_I \sim \frac{1}{|D_k(a_1, \dots, a_N)|^a |D_{k+1}(a_1, \dots, a_N)|^b}.$$

Here $a = 1 - (N - k - 1)\delta$ and $b = (N - k)\delta - 1$ are strictly positive numbers. If we further assume that $\delta = u/v$ is a rational number, then

$$(1.12) \quad Q_I \sim \frac{1}{|\Phi(a_1, \dots, a_N)|^{\delta'}}$$

where $\Phi \in \mathbf{Z}[a_1, \dots, a_N]$ is the polynomial $\Phi = D_k^{av} D_{k+1}^{bve(k)/e(k+1)}$ and $\delta' = 1/ve(k)$. In particular, the size of Q_I can be expressed entirely in terms of polynomials in the coefficients a_j of P , in the range indicated for δ .

Remark 1. The polynomials D_r are defined as follows: Let $F(X)$ be the monic polynomial whose roots are $\{\prod_{\nu=1}^r (y_{i_\nu} - y_{j_\nu})\}$, where the index set is as in (1.7). Then $F(X) = X^M + \sigma_1 X^{M-1} + \cdots + \sigma_M$, and the $\sigma_i = s_i(a_1, \dots, a_N)$ where $s_i(A_1, \dots, A_N) \in \mathbf{Z}[A_1, \dots, A_N]$ (this is just the fact that any symmetric function of the roots is a polynomial in the elementary symmetric functions). Now Δ_r is,

by definition, the absolute value of the largest root of F . Thus

$$\begin{aligned} \Delta_r &\sim \sum |\sigma_j|^{1/j} = \sum (\sigma_j^2)^{1/2j} = \sum (\sigma_j^{2m(j)})^{1/e} \sim \left(\sum \sigma_j^{2m(j)} \right)^{1/e} \\ &= D_r(a_1, \dots, a_N)^{1/e}. \end{aligned}$$

Here the first similarity is elementary (see (2.13) for a proof). The first equality follows from the assumption that P is real valued. The second equality defines $m(j)$, the last similarity is obvious and the last equality defines D_r .

Remark 2. In Theorems 1 and 2 we assume that the polynomial P is monic: This restriction is imposed for simplicity, and can be easily removed. For example, in Theorem 2, if P is not monic, then we can write $P = aP^*$ where P^* is monic, so that $Q_I(P) = |a|^{-\delta} Q_I(P^*)$. Thus the estimate for $Q_I(P^*)$ implies a corresponding estimate for $Q_I(P)$.

In practice, the polynomial $P(y)$ can often be replaced by the $2N$ -degree polynomial $P(y)\overline{P(y)}$ if it is complex valued. With δ replaced simultaneously by $\delta/2$, the condition $\delta N < 2$ is for example unchanged. Thus the requirement that $P(y)$ be real-valued in (b) of Theorem 2 is not a severe restriction. More significantly, simple estimates of the form (1.10) no longer hold when $\delta > \frac{2}{N}$, as examples readily show (cf. Section II). This is also the range where the exponents in (1.11) can become negative, leading to more subtle issues of stability for rational functions. Remarkably, the range $0 < \delta < \frac{2}{N}$ suffices to deal with the stability of the integrals (1.2) in two and three dimensions. In general, it may be necessary to have substitutes of (1.10) for rational functions. The following is a basic example.

Let $P(y)$ be the polynomial (1.3) considered previously, and let $A(y) = \prod_{j=1}^M (y - z_j) = \sum b_j y^{M-j}$ with $z_j \in \mathbf{C}$ be another monic polynomial of degree M . Assume that the interval I is admissible with respect to the product AP and let $\epsilon > 0$. We consider estimates for the integral

$$(1.13) \quad Q(I) = Q = \int_I \frac{|A(y)|^\epsilon}{|P(y)|^\delta} dy .$$

The notion \sim of equivalence in size is now defined up to constants depending only on M, N, δ and ϵ . We have then the following theorem, which says in effect that, in a suitable range, the exact size of Q can again be expressed in terms of the coefficients of A and P :

THEOREM 3. (a) *Assume that $\delta < 1/N$. Then*

$$(1.14) \quad Q \sim |I|^{1-N\delta} |I|^{M\epsilon} .$$

(b) *Assume that $1/N < \delta < 1/(N - 1)$. Assume further that $a_1 = 0$ (this can*

always be achieved after an affine change of variables). Fix $B > 0$ and assume that $I \subseteq [-B, B]$. Then

$$(1.15) \quad 1 + \mathcal{Q} \sim 1 + \frac{\sum_{j=1}^M |b_j L_0^{M-j}|^\epsilon}{L_0^{N\delta-1}}.$$

Here the implied constant depends on M, N, δ, ϵ and B .

The proof of Theorems 1–3 will be given in Section II. In Section III, we illustrate the use of Theorem 1 by deriving an estimate for the distribution function of a function f of two variables in terms of the Newton diagram of f (Theorem 4). An important aspect of the argument is a good choice of coordinate system, a feature which had occurred earlier in the work of Varchenko [23] on two-dimensional oscillatory integrals. Here, the precise property of the coordinate system which we need is that the number of roots of the function f with the same leading term in their Puiseux series expansion be always less than or equal to the inverse of the Newton decay rate (we refer to Theorem 5 below for the complete statement). We show how to construct such a coordinate system. In Section IV, we discuss the applications of our one-dimensional estimates to the issue of stability of the integrals (1.2). Our main results are described in Theorems 6, 7 and 8, which deal respectively with the cases of dimension 2, dimension 3, and of higher dimensions. In dimensions greater or equal to 3, it is known, through Varchenko's example, that the estimate (1.1) is not stable under arbitrary small deformations of f , at least for δ close to the largest values for which (1.1) may hold. It is a very important problem to determine the ranges of δ where some stability can be guaranteed. Theorem 7, which treats the case $n = 3$, and Theorem 8, which establishes the existence of such ranges in higher dimensions and provides a first estimate for them, may be viewed as a step in this direction.

II. Proof of Theorems 1–3. In this section, we establish the one-dimensional estimates stated in Theorems 1–3.

Proof of Theorem 1. The proof of Theorem 1 is an analogue of the proof of the sharp van der Corput Lemma in [15]. The real line \mathbf{R} can be covered by N intervals I_j , defined by

$$(2.1) \quad I_j = \{y \in \mathbf{R}; |y - y_j| \leq |y - y_k| \text{ for all } k\}$$

(I_j is an interval, possibly infinitely extended, since the condition in (2.1) defines in \mathbf{C} an intersection of half-planes). Let S be any subset of $\{1, 2, \dots, N\}$ containing j . For y in the interval I_j , we have

$$(2.2) \quad |y_j - y_k| \leq |y_j - y| + |y - y_k| \leq 2|y - y_k|$$

and thus the function $P(y)$ can be bounded by

$$(2.3) \quad |P(y)| \geq 2^{-N+|S|} \prod_{k \notin S} |y_j - y_k| \prod_{l \in S} |y - y_l|.$$

It follows that the set $I_j \cap \{y; |P(y)| \leq \epsilon\}$ is included in the set

$$(2.4) \quad \left\{ y \in \mathbf{R}; \prod_{l \in S} |y - y_l| \leq C \frac{\epsilon}{\prod_{k \notin S} |y_j - y_k|} \right\}.$$

On the other hand, it can be shown that

$$(2.5) \quad |\{x \in I; |g(x)| \leq \epsilon\}| \leq C_N \epsilon^{1/N}$$

for all $C^{(N)}$, real-valued, functions g on any interval I in \mathbf{R} satisfying $|g^{(N)}| \geq 1$ throughout I . Here C_N is a constant depending only on N . A proof of (2.5) may be found in [2]. Applying this inequality with $g(y)$ given by the real-part of the $|S|$ -th order polynomial $(|S|!)^{-1} \prod_{l \in S} (y - y_l)$, we obtain at once

$$(2.6) \quad |I_j \cap \{y; |P(y)| \leq \epsilon\}| \leq C_N \left(\frac{\epsilon}{\prod_{k \notin S} |y_j - y_k|} \right)^{1/|S|}.$$

Taking the infimum over all clusters S and the supremum over all roots y_j gives Theorem 1.

Proof of Theorem 2. It is convenient to divide the proof into a number of claims.

Claim 1.

(A) If $\delta < 1/N$, then $Q_I \sim |I|^{1-N\delta}$ for all admissible intervals I .

(B) Let P_1, \dots, P_l be monic polynomials of degrees N_1, \dots, N_l . Fix $\delta_1, \delta_2, \dots, \delta_l > 0$ with $\sum_{i=1}^l \delta_i N_i < 1$. Assume that I is P_i -admissible for all i . Then

$$(2.7) \quad \int_I \prod_{i=1}^l |P_i(y)|^{-\delta_i} dy \sim |I|^{1-\sum N_i \delta_i}.$$

Proof. Part (A): We may assume $|I|$ is finite. After making an affine change of coordinates, we may assume that $I = [0, 1]$. But then Q is a continuous function of the y_i on the compact set $L_0 \leq 1$, $Re(y_i) \in [0, 1]$, and thus Q is bounded above and below by positive constants. The proof of part (B) is identical.

Claim 2. If $\delta > 1/N$, then $Q_I \sim Q_J$ if I and J are admissible.

Proof. We may assume that $I = (-\infty, \infty)$ and that $J = [-1, 1]$ (make an affine change of variables). Now if $y \in J$ then clearly $|y - y_i| < |(2 - y) - y_i|$ for all i . Thus

$$Q_J > \int_1^3 |P(y)|^{-\delta} dy .$$

Moreover,

$$\int_3^\infty |P(y)|^{-\delta} dy < \int_3^\infty (y - 1)^{-N\delta} dy = \frac{2^{1-N\delta}}{N\delta - 1} < \frac{2Q_J}{N\delta - 1} .$$

This proves Claim 2.

Claim 3. Assume $1/N < \delta < 1/(N - 1)$. Then $Q_I \sim 1/L_0^{N\delta-1}$ where L_0 is defined as in (1.5–1.6).

Proof. Using Claim 2, we may assume $|I| = L_0$. After making an affine change of variables, we may assume $I = [0, 1]$. But then Q_I is a continuous function of the variables $\{y_1, \dots, y_N\}$ on the compact set $L_0 = 1$, $\operatorname{Re}(y_i) \in [0, 1]$, and is thus bounded above and below by positive constants.

We can now establish the first statement (a) in Theorem 2 by induction on N . The estimate (1.9) is just Part (A) of Claim 1. The estimate (1.10) is true when $N = 2$, for this forces $k = 0$ which is proved in Claim 3. Thus we assume $N > 2$ and $k > 0$. For induction purposes, it is convenient to indicate explicitly the degree N of the polynomial $P(y)$ and the dependence of $Q(y_1, \dots, y_N)$ on the polynomial $P(y)$. Thus we write $P(y) = P_N(y)$, and $Q_I(y_1, \dots, y_N) = Q_I(P_N)$. By Claim 2, we may assume that $|I| = L_0$. If there exists i such that $\operatorname{Im}(y_i) = L_0$, then we have $Q_I(P_N) \sim L_0^{-\delta} Q_I(P_{N-1})$ where P_{N-1} is obtained from P_N by deleting the i^{th} factor. In this case we can proceed by induction.

If $\operatorname{Im}(y_i) < L_0$ for all i then we let $r_i = \operatorname{Re}(y_i)$ and we order the roots in such a way that $r_i \leq r_{i+1}$ for all i . By Claim 2, we may assume that $I = [r_1, r_N]$. Choose ν so that $r_{\nu+1} - r_\nu \geq r_{j+1} - r_j$ for all j . Let $I_1 = [r_1, (r_\nu + r_{\nu+1})/2]$ and $I_2 = [(r_\nu + r_{\nu+1})/2, r_N]$. Let N_ρ be the number of i such that $r_i \in I_\rho$, $\rho = 1$ or 2 . Thus $N = N_1 + N_2$. Then we have

$$(2.8) \quad Q_I(P_N) \sim L_0^{-\delta N_2} Q_{I_1}(P_{N_1}) + L_0^{-\delta N_1} Q_{I_2}(P_{N_2}),$$

where $P_{N_\rho}(y) = \prod (y - y_i)$ and the product is taken over S_{N_ρ} , the set of all i such that $r_i \in I_\rho$. Without loss of generality, we may assume $N_1 \leq N_2$. This implies

$$(2.9) \quad L_0 \sim L_1 \sim \dots \sim L_{N_1-1} .$$

Now we must have $N_2 > k$ (otherwise, $N \leq 2k$, a contradiction). Thus there are

two cases to consider:

Case 1. $N_1 \geq k + 1$ and $N_2 \geq k + 1$. In this case, $\delta < 1/(N_1 + (N_2 - (k + 1))) \leq 1/N_1$. Similarly, $\delta < 1/N_2$. Thus, applying Claim 1 to the right-hand side of (2.8), we obtain $Q_I \sim 1/L_0^{N\delta-1}$. Now we use (2.9) to obtain the result.

Case 2. $N_1 \leq k < N_2$. We still have $\delta < 1/N_1$, so we may argue as before to conclude that the first summand in (2.8) has size $1/L_0^{N\delta-1}$. To estimate the second summand: The fact that $2k < N$ implies that $2(k - N_1) < N_2$ and so we can use induction, with the pair (N, k) replaced by $(N_2, k - N_1)$. Fix r in the range $N_1 \leq r \leq k$. Then if $S \subseteq S_N$ is such that $|S| = N - r$, then $|S| = N - r \geq N - k > N_1$. Hence $L(S) \sim L_0$ unless $S \subseteq S_{N_2}$. This implies that $L_r(S) = L_{r-N_1}(S_{N_2})$. Now we use induction to estimate the second summand in (2.8):

$$\begin{aligned} L_0^{-\delta N_1} Q_{I_2}(P_{N_2}) &\sim \frac{1}{L_0^{\delta N_1} (L_0(S_{N_2}) \cdots L_{(k-N_1-1)}(S_{N_2}))^\delta L_{(k-N_1)}(S_{N_2})^{(N_2-(k-N_1))\delta-1}} \\ &\sim \frac{1}{L_0^{\delta N_1} (L_{N_1} \cdots L_{k-1})^\delta L_k^{(N-k)\delta-1}}. \end{aligned}$$

Combining this with (2.9) completes the inductive step. The estimate (1.10), and hence part (a) of Theorem 2 is proved.

To establish the second statement (b) in Theorem 2, we need to rewrite the cluster scales L_k in terms of the discriminants Δ_r . The precise relation between the two notions is the following, where again, for the purpose of induction on N , we have indicated explicitly the dependence of a set S of N elements by $S = S_N$, and the dependence of Δ_r on S by $\Delta_r = \Delta_r(S_N)$:

Claim 4. Assume that S_N is closed under complex conjugation (i.e., assume the polynomial $p(y) = \prod_{y_i \in S_N} (y - y_i)$ is real valued). If $1 \leq r \leq N/2$ then

$$(2.10) \quad \Delta_r \sim L_0 L_1 \cdots L_{r-1}.$$

Proof. If $N = 2$ the statement is true. We proceed by induction: Assume $N > 2$. If there exists i such that $Im(y_i) = L_0$ then $L_0 = L_1$ and the claim is clearly true when $r \leq 2$. If $r \geq 3$ then let S_{N-2} denote the set S_N with y_i and its complex conjugate removed. Note that for $0 \leq i \leq r - 2$, we have $L_i(S_{N-2}) \sim L_{i+2}(S_N)$. Now we can proceed by induction:

$$\Delta_r(S_N) \sim L_0(S_N)^2 \Delta_{r-2}(S_{N-2}) \sim L_0(S_N)^2 \prod_{i=0}^{r-2} L_i(S_{N-2}) \sim L_0(S_N) L_1(S_N) \prod_{i=2}^r L_i(S_N).$$

If $Im(y_i) < L_0$ for all i , then define S_{N_1} and S_{N_2} as in the inductive part of the

proof of the estimate (1.11). Again we assume $N_1 \leq N_2$. We must have $N_2 \geq r$ (otherwise $N_1 \leq N_2 < r$ which implies $N < 2r$, a contradiction). Two cases remain:

Case 1. $N_2 \geq N_1 \geq r$. Then $\Delta_r \sim L_0^r$. But this is what we need to prove, by virtue of (2.9).

Case 2. $N_2 \geq r > N_1$. Let $M = (i_1, \dots, i_r, j_1, \dots, j_r)$ be an r -tuple which achieves the maximum in the definition of Δ_r . We may assume that M is chosen in such a way that $\{i_\nu, j_\nu\} \cap S_{N_1} \neq \emptyset$ implies $i_\nu \in S_{N_1}$. We claim that $S_{N_1} \subseteq \{i_1, \dots, i_r\}$. If not, then there exists $i \in S_{N_1}$ such that $i \notin \{i_1, \dots, i_r\}$. But $r > N_1$ implies that there exists ν , $1 \leq \nu \leq r$, such that $j_\nu \in S_{N_2}$. Replacing i_ν by i shows that M was not maximal, a contradiction.

Thus we have shown that

$$(2.11) \quad \Delta_r(S_N) \leq C \cdot L_0^{N_1} \Delta_{(r-N_1)}(S_{N_2})$$

for some $C > 0$ depending only on N and δ . Now $2(r - N_1) \leq N_2$, so, by induction,

$$(2.12) \quad \Delta_{(r-N_1)}(S_{N_2}) \sim L_0(S_{N_2}) \cdots L_{(r-N_1-1)}(S_{N_2}) \sim L_{N_1} \cdots L_{r-1}.$$

Moreover, by (2.9), $L_0^{N_1} \sim L_0 L_1 \cdots L_{N_1-1}$ and $L_k(S_{N_2}) = L_{k+N_2}$ if $k < N_2$. Thus we need only show that the inequality in (2.11) is actually an equivalence. But this follows from the fact that $N_2 - 2(r - N_1) \geq N_1$, so that when we pick a maximizing set from S_{N_2} , there are at least N_1 points left over. These N_1 points can be paired with the N_1 points in S_{N_1} to give the required bound.

Assembling the previous results gives part (b) of Theorem 2.

Finally, we turn to the proof of part (c). We begin by observing that

$$(2.13) \quad \sup_{1 \leq i \leq N} |y_i| \sim \sum_{j=1}^N |a_j|^{1/j}.$$

Indeed, the a_j are homogeneous polynomials of degree j in the y_i 's, and after scaling, we may assume that $\sup |y_i| = 1$. Evidently, all the $|a_j|$ are then bounded above by a constant that only depends on N . But their maximum is bounded below as well. To see this, we compare $P(y)$ to $P_0(y) = y^N$: We have $|P(y) - P_0(y)| < |a_1| + \cdots + |a_N|$ if $|y| = 1$. Thus, if $|a_1| + \cdots + |a_N| < 1$, Rouché's Theorem would imply that all N roots of $P(y) = 0$ are inside the open disk of radius one, centered at the origin. But we are assuming that $\sup |y_i| = 1$. Hence, we must have $|a_1| + \cdots + |a_N| \geq 1$. The estimate (2.13) is established.

We can now construct explicitly the polynomials $D_r(a_1, \dots, a_N)$ of part (b)

of Theorem 2. Let M be given as in part (b), and consider the M element set

$$R = \left\{ \prod_{\nu=1}^r (y_{i_\nu} - y_{j_\nu}) \right\}$$

where $(i_1, \dots, i_r, j_1, \dots, j_r) \in \{1, 2, \dots, N\}^{2r}$ runs over all $2r$ -tuples with *distinct* entries. Let \mathcal{P} be the monic polynomial of degree M whose roots are the elements of R , and let U_m be its m^{th} coefficient. Then the U_m are (real valued) symmetric polynomials in the y_i , and hence can be expressed as polynomials in the a_j with integer coefficients. From (2.13) we see that

$$(2.14) \quad \Delta_r \sim \sum_{m=1}^M |U_m|^{\frac{1}{m}} = \sum_{m=1}^M (U_m^{2e(r,m)})^{\frac{1}{e(r)}} \sim (D_r)^{\frac{1}{e(r)}} = |D_r|^{\frac{1}{e(r)}}$$

where $e(r, m) = e(r)/2m \in \mathbf{Z}$ and

$$D_r = \sum_{m=1}^M U_m^{2e(r,m)} .$$

The proof of Theorem 2 is now complete.

We pause for some remarks on the allowed range $0 < \delta < \frac{2}{N}$ for N even, $0 < \delta < \frac{2}{N-1}$ for N odd. A close inspection of the proof of Theorem 2 shows readily that this range of δ cannot be extended in general. More concretely, choose $N = 6$, and consider configurations of 6 points $y_1 < y_2 < y_3 < y_4 < y_5 < y_6$ on the real axis, with the middle interval $|y_3 - y_4|$ much larger than all the other distances. Then the cluster scales L_0, L_1 , and L_2 are all of order $L_0 \sim L_1 \sim L_2 \sim |y_3 - y_4|$. The cluster scale L_3 is as yet undetermined, and can be either $|y_1 - y_3|$ or $|y_4 - y_6|$. Similarly, the shortest cluster scale L_4 is undetermined, and is the smaller one of the two distances m and \tilde{m} , defined respectively to be the shortest distance between points in the two groupings $\{y_1, y_2, y_3\}$ and $\{y_4, y_5, y_6\}$ of roots. On the other hand, breaking the integral Q_I over an admissible interval I into two integrals over respectively $I \cap (-\infty, \frac{y_3+y_4}{2}]$ and $I \cap [\frac{y_3+y_4}{2}, +\infty)$, we obtain at once

$$Q_I \sim L_0^{-3\delta} |y_1 - y_3|^{-\delta} m^{1-2\delta} + L_0^{-3\delta} |y_4 - y_6|^{-\delta} \tilde{m}^{1-2\delta}$$

for $\frac{1}{3} < \delta < \frac{1}{2}$. Clearly this estimate cannot be expressed under the form (1.10) for all configurations of y_1, \dots, y_6 .

We also note that the range $0 < \delta < \frac{2}{N}$ is the natural range where the cluster scales L_k can be expressed in terms of the discriminants Δ_r (cf. (1.7)).

Proof of Theorem 3. Again, it is convenient to break up the proof into a series of simpler statements.

Claim 1. Assume that $\delta < 1/N$. Then

$$\mathcal{Q} \sim |I|^{1-N\delta+M\epsilon}.$$

Proof. As in the case where $A = 1$, we may assume that $I = [0, 1]$ and then, as before, the integral is bounded above and below by positive constants.

Claim 2. Fix $B > 0$. Assume that $1/N < \delta < 1/(N-1)$ and that $I \subseteq [-B, B]$. Then

$$(2.15) \quad 1 + \mathcal{Q}(I) \sim 1 + \mathcal{Q}(J) \sim 1 + \frac{\prod_{j=1}^M (|z_j| + L_0)^\epsilon}{L_0^{N\delta-1}}$$

where $J = [-L_0, L_0]$.

Proof. We break the interval I into three pieces: One piece is the interval $J = [-L_0, L_0]$. The piece to the right of J is denoted by I^+ and the piece to the left by I^- . Then we have $\mathcal{Q} = \mathcal{Q}(I^-) + \mathcal{Q}(J) + \mathcal{Q}(I^+)$. First we prove the second equivalence which is asserted in (2.15). By making a linear change of variables, we may assume that $L_0 = 1$ (note that we may no longer assume that $I \subseteq [-B, B]$: This assumption will only be used to prove the first equivalence). We write $\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2$ where

$$\mathcal{Q}_1(y) = \prod_{|z_j| < 2} (y - z_j) \quad \text{and} \quad \mathcal{Q}_2(y) = \prod_{|z_j| \geq 2} (y - z_j).$$

Note that on the interval J , we have $|\mathcal{Q}_2(y)| \sim \prod (|z_j| + 1)$ where the product is taken over all z_j such that $|z_j| \geq 2$. Therefore

$$\mathcal{Q}(J) = \int_{-1}^1 |A(y)|^\epsilon |P(y)|^{-\delta} dy \sim \prod_{|z_j| \geq 2} (|z_j| + 1)^\epsilon \int_{-1}^1 |\mathcal{Q}_1(y)|^\epsilon |P(y)|^{-\delta} dy.$$

Now we simply observe that the integral on the right side is bounded above and below by positive constants, and thus $\mathcal{Q}(J)$ has the size predicted for \mathcal{Q} . Thus, to complete the claim, we need to show $\mathcal{Q}(I^+) < \mathcal{Q}(J)$ and $\mathcal{Q}(I^-) < \mathcal{Q}(J)$.

It suffices to estimate $\mathcal{Q}(I^+)$: The size of $\mathcal{Q}(I^+)$ is given by

$$\int_{L_0}^B \frac{|A(y)|^\epsilon}{|P(y)|^\delta} dy \sim \int_{L_0}^B \frac{\prod |y - z_j|^\epsilon}{y^{N\delta}} dy < \int_{L_0}^B \frac{\prod (y + |z_j|)^\epsilon}{y^{N\delta}} dy \sim \int_{L_0}^B \frac{\prod (y^\epsilon + |z_j|)^\epsilon}{y^{N\delta}} dy.$$

Letting S_j denote the j^{th} symmetric polynomial in M variables we obtain:

$$\mathcal{Q}(I^+) < C_1 \sum_{k=0}^M \int_{L_0}^B \frac{y^{k\epsilon} S_{M-k}(|z_j|^\epsilon)}{y^{N\delta}} dy < C_2 \sum_{k=0}^M S_{M-k}(|z_j|^\epsilon) \left[1 + \frac{L_0^{k\epsilon}}{L_0^{N\delta-1}} \right]$$

where C_1 and C_2 are positive constants depending only on M, N, δ, ϵ and B . Finally, we note that the last term in the above inequality is bounded by

$$\prod_{j=1}^M (1 + |z_j|)^\epsilon + \frac{\prod_{j=1}^M (|z_j| + L_0)^\epsilon}{L_0^{N\delta-1}}$$

and this completes the proof of Claim 2.

Next we show that

$$(2.16) \quad \prod_{j=1}^M (|z_j| + L_0) \sim \sup_{|\zeta|=1} |A(\zeta L_0)|.$$

Dividing both sides by L_0^M and replacing z_j by z_j/L_0 , we may assume that $L_0 = 1$. Thus we must show that

$$\prod_{j=1}^M (|z_j| + 1) \sim \sup_{|\zeta|=1} \prod_{j=1}^M |z_j - \zeta|$$

The triangle inequality shows that the right side is less than or equal to the left side.

Now, for $z_j \neq 0$ let $\zeta_j = z_j/|z_j|$. The ζ_j 's partition the unit circle into at most M intervals. Let ζ be the midpoint of the longest interval. Then $|z_j - \zeta| \geq |z_j| - 1 \sim |z_j| + 1$ provided $|z_j| \geq 2$. If $|z_j| < 2$ then $|z_j - \zeta|$ is bounded below by a positive constant which depends only on M . This completes the proof of (2.16). To finish the proof of Theorem 3 we need to show that

$$\sup_{|\zeta|=1} |A(\zeta L_0)| \sim \sum_{j=0}^M |b_j L_0^{M-j}|.$$

Replacing z by zL_0 , we may assume that $L_0 = 1$. Then the triangle inequality shows that $|A(\zeta)| \leq \sum_{j=0}^M |b_j|$. On the other hand, the Cauchy integral formula shows that $|b_j| \leq (2\pi)^{-1} \sup_{|\zeta|=1} |A(\zeta)|$. This completes the proof of Theorem 3.

III. Growth of real-analytic functions. The purpose of this section is to describe some basic properties of the distribution function of real-analytic functions f of two variables, together with the local integrability of $|f|^{-\delta}$. Many

cases of the estimates that we describe are known to experts and can be derived, for example, from the results of Varchenko on oscillatory integrals (for the sake of completeness, we shall also include a proof along these lines at the end of this section). However, they have not, to our knowledge, been written down explicitly in the literature, and we shall take the opportunity to illustrate the use of our one-dimensional estimates in this context.

THEOREM 4. *Let f be a real-analytic function in a neighborhood of the origin in \mathbf{R}^2 . Then there is an analytic coordinate system and a small neighborhood V of the origin in which the following holds*

(a) $|\{x \in V; |f| < \epsilon\}| \leq C_f \epsilon^\nu$, for any $\nu < \delta$, where δ is the Newton distance of f in the coordinate system. (A formula for the Newton distance can be found in (3.8) below).

(b) In particular, we have

$$(3.1) \quad \int_V |f|^{-\nu} < \infty$$

for any $\nu < \delta$.

We divide the proof into several steps:

Step 1. Let (x, y) denote a coordinate system near the origin in \mathbf{R}^2 , which will be suitably chosen later. Then

$$(3.2) \quad |\{|f| < \epsilon\}| \sim \iint_{|f| \leq \epsilon} dx dy = \int |\{y; |f(x, y)| \leq \epsilon\}| dx.$$

Thus our task is to find estimates for $|\{y; |f(x, y)| \leq \epsilon\}|$ which are integrable in x . For this we need the following:

Step 2. In view of the Weierstrass preparation theorem, the function f can be expressed, up to a nonvanishing factor which we ignore, as a polynomial in y of some degree N , with coefficients analytic in x . Factoring out this polynomial, we can express f as

$$(3.3) \quad f(x, y) = x^\alpha y^\beta \prod_\nu (y - r_\nu(x))$$

where α, β are nonnegative integers, and $r_\nu(x)$ are Puiseux series in x . Let a_ν be the collection of leading exponents in the Puiseux series $r_\nu(x)$, i.e., $r_\nu(x) = c_\nu x^{a_\nu} + O(x^{A_\nu})$, where $c_\nu \neq 0$ and $A_\nu > a_\nu$. By restricting ourselves to a small enough neighborhood of the origin, we may assume that $a_\nu > 0$, since the factor $y - r_\nu(x)$ may otherwise be absorbed in a nonvanishing factor. We order the

distinct exponents a_ν into an increasing list of exponents a_l

$$0 < a_1 < a_2 < \dots < a_n,$$

and define

$$(3.4) \quad m_l = \#\{r_\nu; r_\nu(x) = c_\nu x^{a_l} + \dots \text{ for some } c_\nu\}.$$

We shall also need the following finer subdivision of the roots in (3.4). Let k index the set c_{lk} of *distinct* coefficients c_ν occurring in (3.4). Then $m_l = \sum_k m_{lk}$, with m_{lk} defined by

$$(3.5) \quad m_{lk} = \#\{r_\nu; r_\nu(x) = c_{lk} x^{a_l} + \dots \text{ for some } c_{lk}\}.$$

To lighten the notation, we shall assume that $\alpha = \beta = 0$, and only indicate at the end the routine modifications needed for the general case.

The Newton diagram of f is easily worked out. Its vertices are at the points

$$(m_1 + \dots + m_N, 0), (m_2 + \dots + m_N, a_1 m_1), (m_3 + \dots + m_N, a_1 m_1 + a_2 m_2), \dots$$

The equations of the faces of the Newton diagram are

$$q = -\frac{1}{a_l} p + \frac{1}{a_l} (a_1 m_1 + \dots + a_{l-1} m_{l-1}) + m_l + \dots + m_N.$$

Thus their prolongations intersect the diagonal $q = p$ at the points $(\delta_l^{-1}, \delta_l^{-1})$, with

$$(3.6) \quad \delta_l^{-1} = \frac{A_{l-1} + a_l B_{l-1}}{1 + a_l} = \frac{A_l + a_l B_l}{1 + a_l}$$

where we have introduced the notation

$$(3.7) \quad A_l = a_1 m_1 + \dots + a_l m_l = \sum_{j \leq l} m_j a_j, \quad B_l = m_{l+1} + \dots + m_N = \sum_{j \geq l+1} m_j.$$

Recall that the Newton distance δ is defined by

$$(3.8) \quad \delta = \min_l \delta_l = \min_l \frac{1 + a_l}{A_l + a_l B_l}.$$

Thus the Newton diagram encodes the following information on the roots of $f(x, y)$. The set of roots with a given leading exponent x^{a_l} corresponds to the face l with slope $-\frac{1}{a_l}$. The vertical drop of that face gives the generalized multiplicity m_l . Each face also gives rise to a homogeneous polynomial f_l , obtained by keeping in the Taylor expansion of $f(x, y)$ only the terms whose exponents lie on the given

face. The zeroes $y = c_{kl}x^{a_l}$ of f_l are in turn the leading coefficients of the zeroes of the original function $f(x, y)$. We recall that *the main face* is the face of the Newton diagram which meets the diagonal $q = p$.

Step 3. Our strategy is to apply the estimate (1.4) to the polynomial $f(x, y)$, viewed as a function of y . Near each root $r_\nu(x) = c_\nu x^{a_l} + \dots$, we wish to obtain the optimal estimate for $|\{y; |f(x, y)| \leq \epsilon\}|$ among all estimates integrable in x near the origin. For this, we have to select a suitable cluster S of roots to be removed as in (1.4). Ignoring for the moment the fact that δ_l^{-1} may not be an integer, and that the estimate claimed in Theorem 4 holds not for δ itself, but only for any $\delta^l < \delta$, it is clear that we should remove a cluster $S = \delta_l^{-1}$ of roots. This is possible only if the multiplicity of the roots $r_\nu(x)$ is less than δ_l^{-1} . More generally, the estimate (2.3) provides useful information only when the distance $|r_\nu(x) - r_\mu(x)|$ between $r_\nu(x)$ and the remaining roots $r_\mu(x) \notin S$ can be evaluated. In particular, the easy and precise estimate

$$(3.9) \quad |r_\nu(x) - r_\mu(x)| \sim |x|^{\min(a_\nu, a_\mu)}$$

is available only if we remove all roots $r_\mu(x)$ with the same leading exponent *and* the same leading coefficient c_{lk} as $r_\nu(x)$. Thus it is desirable to insure that

$$(3.10) \quad m_{lk} \leq \delta_l^{-1}.$$

This condition (3.10) depends on the choice of coordinate system, since the generalized multiplicity m_l is an intrinsic notion, while δ_l is not. A crucial feature of two dimensions is that we can always construct a suitable coordinate system in which (3.10) does hold. More precisely,

THEOREM 5. *We always have*

$$(3.11) \quad \delta^{-1} \geq m_{lk}$$

when the index l does not correspond to the main face. If the inequality (3.11) is violated for some index m_{lk} with l corresponding to the main face, then either there exists a power series $P(x)$ such that the new function

$$(3.12) \quad \tilde{f}(x, y) = f(x, y - P(x))$$

and its Newton distance $\tilde{\delta}$ always satisfy

$$\tilde{\delta}^{-1} \geq m_{\tilde{l}\tilde{k}}$$

for all $\tilde{m}_{\tilde{l}\tilde{k}}$, or there exists a power series $P(y)$ so that $f(x - P(y), y)$ has the same property.

Proof of Theorem 5. Consider the faces l of the Newton diagram distinct from the main face. Set

$$a_l = \frac{q_l}{p_l}, \quad q_l \text{ and } p_l \text{ relatively prime.}$$

The key observation is that if $y = c_{kl} x^{\frac{q_l}{p_l}} + \dots$ is a root, then so is $\zeta c_{kl} x^{\frac{q_l}{p_l}} + \dots$ for ζ a p_l -th root of unity. In particular,

$$(3.13) \quad m_{lk} \leq \frac{1}{p_l} m_l.$$

On the right of the main face, we conclude that the right hand side of (3.13) is $\leq m_l \leq \delta^{-1}$, since m_l corresponds to the projection of the l face on the vertical q axis, which is clearly smaller than δ_l^{-1} . On the left of the main face, we let M_l and N_l be the lengths of the projections of the face l respectively on the p and the q axes. The segment N_l is actually just equal to m_l . Furthermore, we have

$$\frac{M_l}{N_l} = \frac{q_l}{p_l}.$$

It follows that

$$(3.14) \quad m_{lk} \leq \frac{1}{p_l} m_l = \frac{M_l}{q_l} \leq M_l \leq \delta_l^{-1},$$

proving the first part of Theorem 5. Let l correspond now to the main face, and consider its equation

$$(3.15) \quad \frac{p_l p + q_l q}{q_l + p_l} = \delta^{-1}$$

which is dictated by its slope $-\frac{q_l}{p_l}$, and the fact that it passes by the point $(\delta^{-1}, \delta^{-1})$. Since p_l, q_l are positive, the left-hand side of (3.15), when restricted to any half-line in the first quadrant, is a decreasing function of the distance to the origin. Restricting it to the q -axis, we may write

$$\delta^{-1} \geq \frac{q_l N_l}{q_l + p_l} \geq \frac{p_l q_l}{p_l + q_l} m_{lk},$$

where we also made use of the inequality (3.13). If the inequality (3.11) is violated by the main face, it follows that

$$p_l q_l \leq p_l + q_l,$$

and thus either p_l or q_l is one.

Say $p_l = 1$ and $\delta^{-1} < m_{lk}$ for some k . This implies that there exists k_1, \dots, k_e with $e > \delta^{-1}$ so that the roots y_1, \dots, y_{k_e} have the same leading terms $c_q x^q$. For a polynomial (respectively a power series) $P(x) = c_q x^q + \dots + c_r x^r$ (respectively $P(x) = c_q x^q + \dots +$), we say that $P(x)$ has generalized multiplicity e if there exist e roots agreeing with $P(x)$ up to higher order terms (respectively agreeing exactly with $P(x)$). Choose then $P(x)$ of maximal degree (possibly infinite) such that $P(x)$ occurs with multiplicity e greater than δ^{-1} . Let $\tilde{f}(x, y) = f(x, y - P(x))$, and consider the Newton diagram of \tilde{f} . Then the intersection of the Newton diagrams of f and \tilde{f} consist of

$$\{(v, j); j \geq j^* + e\}$$

where j^* is the y -coordinate of the higher corner of the main face. If $P(x)$ is a power series, then the new main face will be vertical, and its multiplicity equals $\tilde{\delta}^{-1}$. Otherwise, the new main face has a finite slope $\tilde{\mu} > 1$. We claim that $\tilde{\delta}^{-1} \geq \delta^{-1} \geq$ multiplicity of new main face. This is clear if $\tilde{\mu}$ is not an integer (since $\frac{pq}{p+q} \geq 1$ if $q \geq p \geq 2$). If $\tilde{\mu}$ is an integer, then this again must hold, otherwise the maximality of $P(x)$ is contradicted. This proves Theorem 5.

We observe that in higher dimensions, we still do not know how to characterize good coordinate systems with respect to Newton diagrams. Some necessary conditions and conjectures are formulated in [6].

Step 4. Henceforth we assume that we are in a coordinate system where (3.11) holds for all m_{lk} . For each root $r_\nu(x)$, we shall use a convex combination of two estimates of the form

$$\left(\frac{\epsilon}{\prod_{\mu \neq \nu} |r_\nu - r_\mu|} \right)^{\frac{1}{K}}$$

for two suitably chosen values of K . In all cases K will be greater than m_{lk} , so that the estimate (3.9) will apply. Let L is the unique integer defined by

$$(3.16) \quad L \leq \delta_l^{-1} < L + 1.$$

We consider separately the following three cases:

- (i) $L < L + 1 \leq B_{l-1}$;
- (ii) $L \leq B_{l-1} < L + 1$;
- (iii) $B_{l-1} < L < L + 1$.

In case (i), we use the estimates resulting from removing respectively L and $L + 1$ roots, including all the roots with the same leading exponent and coefficient as $r_\nu(x)$, and as many roots as necessary with leading exponent greater or equal

to a_l . The contribution of $r_\nu(x)$ to the estimate for $|\{y; |f(x, y)| \leq \epsilon\}|$ is then bounded by

$$(3.17) \quad \min \left[\frac{\epsilon}{|x|^{A_l+a_l(B_l-L)}} \right]^{\frac{1}{L}}, \left[\frac{\epsilon}{|x|^{A_l+a_l(B_l-L-1)}} \right]^{\frac{1}{L+1}}.$$

We claim that for any $\kappa > 0$ sufficiently small, there exists a θ between 0 and 1 such that

$$(3.18) \quad \left[\frac{\epsilon}{|x|^{A_l+a_l(B_l-L)}} \right]^{\frac{\theta}{L}} \left[\frac{\epsilon}{|x|^{A_l+a_l(B_l-L-1)}} \right]^{\frac{1-\theta}{L+1}} = \frac{\epsilon^{\delta_l-\kappa}}{|x|^{1-\tilde{\kappa}}},$$

where $\tilde{\kappa}$ is a small positive number depending on κ . In fact, θ is determined by the condition

$$\frac{\theta}{L} + \frac{1-\theta}{L+1} = \delta_l - \kappa,$$

and works out to be

$$(3.19) \quad \theta = L(L+1)\delta_l - L - \kappa L(L+1).$$

Since $\delta_l(L+1) > 1$, the parameter θ defined in (3.19) is strictly positive for κ small enough. Similarly, the fact that $\delta_l L \leq 1$ implies that θ is strictly less than 1. Thus we can take a convex combination θ of the two estimates in (3.17). The resulting power of $|x|$ in the denominator is given by

$$(A_l + a_l(B_l - L))\frac{\theta}{L} + (A_l + a_l(B_l - L - 1))\frac{1-\theta}{L+1} = (A_l + a_l B_l) \left(\frac{\theta}{L} + \frac{1-\theta}{L+1} \right) - a_l.$$

In view of (3.8) and (3.19), we find

$$(A_l + a_l B_l)(\delta_l - \kappa) - a_l = (1 + a_l) - \tilde{\kappa} - a_l = 1 - \tilde{\kappa},$$

with $\tilde{\kappa} = \kappa(A_l + a_l B_l)$. This establishes (a) of Theorem 4 in the case (i).

Next, we consider the case (iii). Since B_{l-1} is an integer, the condition $B_{l-1} < L + 1$ actually implies that $B_{l-1} \leq L \leq \delta_l^{-1}$. In view of the expression (3.8) for δ_l , this is equivalent to

$$B_{l-1} < A_{l-1}.$$

As l decreases, the difference $B_l - A_l$ increases, with $B_0 - A_0$ becoming positive and reaching ultimately the value n (recall that we assumed that $\alpha = \beta = 0$ in (3.3)). Let s be the first value satisfying $B_{s-1} \geq A_{s-1}$, $B_s < A_s$. In view of the above, we have $s \leq l - 1$. We make use then of the two estimates obtained by removing B_{s-1} and B_s roots, consisting respectively of all the roots of leading

exponents $\geq a_s$ and $> a_s$ respectively. All the remaining roots clearly have leading exponent strictly less than a_l . The resulting bound is

$$(3.20) \quad \min \left[\frac{\epsilon}{|x|^{A_s}} \right]^{\frac{1}{B_s}}, \left[\frac{\epsilon}{|x|^{A_{s-1}}} \right]^{\frac{1}{B_{s-1}}}.$$

In analogy with case (i), we shall take the convex combination θ of these two estimates corresponding to

$$\frac{\theta}{B_{s-1}} + \frac{1-\theta}{B_s} = \delta_s - \kappa,$$

where κ is a sufficiently small positive number. It follows that $\theta = -m_s^{-1}[B_s B_{s-1} \delta_s - B_{s-1} - \kappa B_s B_{s-1}]$. In particular, the exponent of $|x|$ in the denominator is found to be

$$\begin{aligned} \frac{\theta A_{s-1}}{B_{s-1}} + \frac{(1-\theta)A_s}{B_s} &= -\theta \frac{a_s m_s}{B_{s-1}} + A_s(\delta_s - \kappa) \\ &= a_s(B_s(\delta_s - \kappa) - 1) + A_s(\delta_s - \kappa) \\ &= (A_s + a_s B_s)(\delta_s - \kappa) - a_s \\ &= 1 - \tilde{\kappa} \end{aligned}$$

where $\tilde{\kappa} = \kappa(A_s + a_s B_s)$ is a small positive number. This is the desired bound.

Finally, the case (ii) is essentially a subcase of (iii). Indeed, since both L and B_{l-1} are integers, the condition $L \leq B_{l-1} < L + 1$ just means that $L = B_{l-1}$, and (3.8), (3.16) imply that $B_{l-1} < A_{l-1}$ just as in (iii). We make use of the two estimates obtained by removing respectively B_{l-1} and B_{l-2} . This corresponds to the case $s = l - 1$ treated previously. Thus we have in all cases an estimate of the form

$$(3.21) \quad \min_K \left(\frac{\epsilon}{\prod_{\mu \notin K} |r_\nu - r_\mu|} \right)^{\frac{1}{K}} \leq \sup_s \frac{\epsilon^{\delta_s - \kappa}}{|x|^{1 - \tilde{\kappa}}}$$

for $\kappa, \tilde{\kappa}$ positive and arbitrarily small. Integrating in x gives part (a) of Theorem 4. Part (b) is a trivial consequence of part (a). This establishes Theorem 4 in the case $\alpha = \beta = 0$. The general case is easily established in the same way, by treating the roots corresponding to the factor y^β as having formally leading exponent $a_l = \infty$, and replacing ϵ in our earlier arguments by $\epsilon |x|^{-\alpha}$. The coordinates (A_l, B_l) of the vertices of the Newton diagram just get shifted to $(A_l + \alpha, B_l + \beta)$. The proof of Theorem 4 is complete.

We can relate Theorem 4 to the decay rate of oscillatory integrals, when the distribution function $F(\epsilon) = |\{(x, y) \in V; |f(x, y)| < \epsilon\}|$ of f admits an asymptotic

expansion of the form

$$(3.22) \quad F(\epsilon) \sim \sum_{\alpha\beta} c_{\alpha\beta} \epsilon^\alpha (\log \epsilon)^\beta,$$

where the exponent α ranges over a discrete set of rational numbers, and the range of the exponent β is finite for each α . This is known to hold in many cases, and in particular when f has an isolated minimum [1]. Since we can write

$$(3.23) \quad \int_V |f|^{-s} = \int_0^\infty t^{-s} dF(t),$$

it follows immediately that the left-hand side of (3.23) defines a meromorphic function of s in the entire complex plane, with poles at $s = \alpha$, where α is any of the rational exponents occurring in the asymptotic expansion (3.22). In particular, the largest left half-plane on which it is holomorphic is $\text{Re } s < \alpha_0$, where α_0 is the smallest of all the α exponents occurring with nonvanishing coefficients in (3.22). On the other hand, assume that the estimate (1.1) for oscillatory integrals holds for some positive δ and γ . For each fixed ν , $0 < \nu < 1$, recall the following identity for any real number f

$$(3.24) \quad c_\nu |f|^{-\nu} = \lim_{T \rightarrow \infty} \int_{-T}^T e^{i\lambda f} |\lambda|^{-1+\nu} d\lambda$$

where c_ν is a nonvanishing constant. An integration by parts readily shows that the limit in (3.24) exists, and that the improper integral can be treated for all practical purposes as a converging integral on the whole real line. Thus we can view it as such, and may write, for χ a smooth nonnegative cut-off function which is identically 1 on V

$$(3.25) \quad c_\nu \iint_{\mathbf{R}^2} |f(x, y)|^{-\nu} \chi(x, y) dx dy = \int_{\mathbf{R}} \left[\iint_{\mathbf{R}^2} e^{i\lambda f(x, y)} \chi(x, y) dx dy \right] |\lambda|^{-1+\nu} d\lambda.$$

This shows that the left-hand side of (3.25) is a convergent integral for any $\nu < \delta$. Setting $\nu = \text{Re } s$, we deduce that the left-hand side of (3.23) is a holomorphic function of s in the open half-plane $\text{Re } s < \delta$. In particular $\delta = \alpha_0$, which means that, ignoring logarithmic factors, the leading growth of the distribution function $F(\epsilon)$ is exactly the decay rate of the oscillatory integral with phase $f(x, y)$.

Now Varchenko [23] has shown that, in suitable coordinate systems, the decay rate (1.1) holds with δ given by the Newton decay rate. It is not difficult to check that the coordinate system constructed in Theorem 5 satisfies Varchenko's criteria, and hence (1.1) applies. (Equivalently, we may show directly, by passage to polar coordinates and an application of the one-dimensional van der Corput lemma, that (1.1) holds without logarithmic factors and for any δ strictly less than the Newton rate, which suffices for our purposes.) As noted previously, this

gives the leading term in the asymptotic expansion of $F(\epsilon)$, and the statement (a) of Theorem 4 follows.

IV. Stability of integrals of real-analytic functions. We now come to the main motivation for Theorems 1–3, namely the issue of stability of integrals of the form (1.2). In fact, only Theorem 2 is needed here. Roughly speaking, the bounds from both above and below provided in Theorem 2 allow us to argue as follows. If the integral (1.1) is finite for a function $f(x', y)$ which is a polynomial in y , then the expression (1.11) is a product of negative powers of *polynomials* in the coefficients of f which is integrable in x' . Under small perturbations of f , these polynomials change analytically (unlike the zeroes of $f!$), and the integrability in x' should be preserved. We shall see below that these arguments are powerful enough to give completely general stability theorems in two and three dimensions, together with a theorem in arbitrary dimensions, but with a narrower range of allowable exponents δ .

To state these theorems, we need some notation. Let $r = (r_1, \dots, r_n) \in \mathbf{R}^n$ and assume that $r > 0$ (this is, $r_i > 0$ for all i). For $m \leq n$ denote by $I^{(m)}(r)$ (*resp.* $U^{(m)}(r)$) the closed polydisk in \mathbf{R}^m (*resp.* \mathbf{C}^m) of radius (r_1, \dots, r_m) which is centered at the origin. When $m = n$, we shall sometimes write $I(r)$ (*resp.* $U(r)$) for $I^{(m)}(r)$ (*resp.* $U^{(m)}(r)$). If U is a closed set in \mathbf{C}^n and if f is a complex valued function on U , we let $|f|_U = \sup_{z \in U} |f(z)|$. Finally, if $f(x_1, x_2, \dots, x_n)$ is real analytic in a neighborhood of the origin in \mathbf{R}^n , then we say that N is the order of vanishing (or the *multiplicity*) of f (at the origin) if we can write $f = \sum_{i=N}^{\infty} f_i$ where $f_i(x_1, x_2, \dots, x_n)$ is a homogeneous polynomial of degree i and f_N is not identically zero.

The precise statement of stability in two dimensions reads as follows:

THEOREM 6. *Let f be a complex valued real-analytic function on $I(r) \subseteq \mathbf{R}^2$. Assume that $\int_{I(r)} |f|^{-\delta} < \infty$ for some $\delta > 0$. Then there exists $0 < s < r$ and an $\epsilon > 0$ so that*

$$(4.1) \quad \sup_g \int_{I(s)} |g|^{-\delta} < \infty$$

where the sup is taken over all g , holomorphic on $U(r)$ satisfying $|f - g|_{U(r)} < \epsilon$ (by abuse of notation, we denote by f the unique holomorphic extension of f to $U(r)$).

Using Mellin transforms as we did at the end of Section III, we can also derive Theorem 6 from Karpushkin's Theorem on the stability of oscillatory integrals in two dimensions [8, 9]. However, the proof we give here is much simpler than the proof of Karpushkin's Theorem.

In three dimensions, we have the following stability result:

THEOREM 7. *Let f be a complex valued real-analytic function on $I(r) \subseteq \mathbf{R}^3$. Assume that $\int_{I(r)} |f|^{-\delta} < \infty$ for some $\delta > 0$ such that $\delta < 2/N$, where N is the*

multiplicity of f at the origin. Then there exists $0 < s < r$ and an $\epsilon > 0$ so that

$$\sup_g \int_{I(s)} |g|^{-\delta} < \infty$$

where the sup is taken over all g , holomorphic on $U(r)$ satisfying $|f - g|_{U(r)} < \epsilon$.

The following example in 3 dimensions due to Varchenko [23] shows that the stability range provided by Theorem 7 is sharp. Let the function $f_\epsilon(x_1, x_2, x_3)$ be given by

$$f_\epsilon(x_1, x_2, x_3) = (\epsilon x_1^2 + x_1^4 + x_2^2 + x_3^2)^2 + x_1^{4p} + x_2^{4p} + x_3^{4p},$$

and let δ_0 be the supremum of all δ for which the integral (1.2) over a small neighborhood of the origin would be finite. Then δ_0 is given by $\delta_0 = \frac{5}{8}$ for $\epsilon = 0$, $\delta_0 = \frac{3}{4}$ for $\epsilon > 0$, and $\delta_0 < \frac{1}{2} + \gamma(p)$ for $\epsilon < 0$ and $\gamma(p) \rightarrow 0$ as $p \rightarrow \infty$. In our set-up, the multiplicity N of f_ϵ is $N = 4$, and Theorem 7 explains the emergence of $\frac{2}{N} = \frac{1}{2}$ as the stability range for f_ϵ .

To describe the case of higher dimensions, we introduce the following terminology. A real analytic function f on $I(r)$ is said to be a local N^{th} power if there exists $0 < s < r$ and a real analytic function h on $I(s)$ such that $f|_{I(s)} = h^N$. We have then

THEOREM 8. *Let $f = f(x_1, x_2, \dots, x_n)$ be a complex valued real analytic function in $I(r)$. Let N be the order of vanishing of f at the origin.*

(A) *If $\delta \geq n/N$ then*

$$(4.2) \quad \int_{I(r)} |f|^{-\delta} = \infty .$$

(B) *If $\delta < 1/N$ then*

$$(4.3) \quad \int_{I(r)} |f|^{-\delta} < \infty .$$

Moreover, for some $\epsilon > 0$ we have

$$(4.4) \quad \sup_g \left\{ \int_{I(s)} |g|^{-\delta} : g \text{ holomorphic on } U(r), |g - f|_{U(r)} < \epsilon \right\} < \infty .$$

(C) *If f is a local N^{th} power then for every $\delta \geq 1/N$ we have $\int_{I(s)} |f|^{-\delta} = \infty$ for all s .*

(D) If f is not a local N^{th} power there exists $0 < s < r$ and $\delta > 1/N$ such that:

$$(4.5) \quad \int_{I(s)} |f|^{-\delta} < \infty .$$

Moreover, for some $\epsilon > 0$ condition (4.4) holds.

Remark 1. In the proofs of Theorems 6, 7 and 8, we may assume that f is real valued (and in particular, we may apply (1.11) and (2.10)). To see this, let us define \bar{f} to be the function which is characterized by the formula $\bar{f}(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbf{C}^n$. Then for $x \in \mathbf{R}^n$ we have $|f(x)|^{-\delta} = |f(x)\bar{f}(x)|^{-\delta/2}$, and so f may be replaced by $f\bar{f}$, δ by $\delta/2$ and N by $2N$ (which does not affect the value of $N\delta$; hence the assumptions $N\delta > 1$ and $N\delta < 1$ of Theorem 7 remain invariant).

Remark 2. The value of δ in Theorem 8 D can be made explicit: We first explain how this can be done in a special case, where the notation is simpler, and then we give the general formulation.

Let $z = (x_1, \dots, x_{n-1})$ and $w = x_n$, and assume that $f(x_1, \dots, x_n) = f(z; w)$ is real analytic on $I(r)$. Suppose that f has an expansion of the form

$$f(x_1, \dots, x_n) = f(z; w) = \sum_{i=0}^N u_{N-i}(z)w^i$$

where $f(0; w) = w^N$. Assume that $u_1(z) = 0$ and that $u_m(z) \neq 0$ for some m . Then we can take δ to be any number satisfying $\delta < 1/(N-1)$ and

$$\frac{1}{N} < \delta < \sup_m \left(\frac{1}{N} + \frac{m}{N N'(m)} \right) ,$$

where $N'(m)$ is the order of vanishing of $u_m(z)$ at the origin.

In general, let $N > 0$ and let $\{A_{N-i} : i \geq 0, i \neq N\}$ be a family of independent variables. Then there exist $U_2, U_3, \dots, U_N \in \mathbf{Z}[A_{-1}, A_{-2}, \dots][[A_1, \dots, A_N]]$ (the ring of power series in the variables A_1, A_2, \dots, A_N whose coefficients are polynomials in the variables A_{-1}, A_{-2}, \dots) with the following properties:

(1) Suppose f is a real analytic on $I(r)$, and suppose we can write

$$f(x_1, \dots, x_n) = f(z; w) = \sum_{i=0}^{\infty} a_{N-i}(z)w^i$$

with $f(0; w) = a_0 w^N + O(w^{N+1})$, ($a_0 = a_0(0) \neq 0$). Assume as well that $a_0(z) = 1$ (this can always be achieved by replacing $f(z; w)$ with $f(z; w)/a_0(z)$). Then f is local N^{th} power if and only if $u_m(z) = 0$ for all m , $2 \leq m \leq N$, where

$u_m(z) = U_m(\{a_i(z); i \geq 0, i \neq N\})$ (Thus the U_m may viewed as “generalized discriminants”.)

(2) We can compute δ in terms of the U_m as follows: If f is not a local N^{th} power then we can take δ to be any number satisfying $\delta < 1/(N - 1)$ and

$$\frac{1}{N} < \delta < \sup_m \left(\frac{1}{N} + \frac{m}{N} \frac{1}{N'(m)} \right) ,$$

where $N'(m)$ is the order of vanishing of $u_m(z)$ at the origin.

(3) The $u_m(z)$ can be characterized as follows: Let $f(z; w) = u(z; w)p(z; w)$ be the Weierstrass factorization of f into a product of a nowhere vanishing function u , and a Weierstrass polynomial

$$p(z; w) = p_N(z) + p_{N-1}(z)w + \dots + p_1(z)w^{N-1} + w^N .$$

Then the $u_m(z)$ are the coefficients of the polynomial $p(z; w - p_1/N)$:

$$p(z; w - \frac{p_1}{N}) = u_N(z) + u_{N-1}(z)w + \dots + u_2(z)w^{N-2} + w^N .$$

We turn now to the proofs of Theorems 6, 7 and 8, as well as of the subsequent remarks. Common to the proofs of all three theorems is a reduction to the case of polynomials in one of the variables, using the Weierstrass preparation theorem. Since we must deal with perturbations of f and reduce them also to polynomials at the same time, we need the following version of “Weierstrass preparation with parameters,” which shows that the polynomial $p = p(f)$ associated to f varies continuously with respect to f in the sup norm.

Step 1. Weierstrass preparation with parameters.

LEMMA 1. *Let $f(z_1, z_2, \dots, z_{n-1}, w) = f(z; w)$ be holomorphic on $U^{(n)}(r) \subseteq \mathbb{C}^n$ for some $r > 0$. Assume that $f(0; w) = aw^N + \dots$ higher order terms, with $a \neq 0$. Then there exists $\epsilon > 0$ and $0 < s < r$ with the following property: If g is holomorphic on $U(r)$ and if $|f - g|_{U(r)} < \epsilon$ then $g|_{U(s)}$ can be written uniquely as*

$$(4.6) \quad g(z; w) = u_g(z; w)p_g(z; w)$$

where $u_g(z; w)$ is nowhere vanishing and holomorphic on $U^{(n)}(s)$, and $p_g(z; w)$ is a Weierstrass polynomial in w of degree N , that is,

$$p_g(z; w) = \sum_{i=0}^N a_{i,g}(z)w^{N-i}$$

with $a_i(z) = a_{i,g}(z)$, holomorphic on $U^{(n-1)}(s)$ and $a_0 = 1$.

Moreover, the assignment of p to g is continuous at f : If $\alpha > 0$ is given then there exists $\epsilon > 0$ such that $|f - g|_{U(t)} < \epsilon$ implies $|p_f - p_g|_{U(s)} < \alpha$, $|u_f - u_g|_{U(s)} < \alpha$, $|1 - u_f/u_g|_{U(s)} < \alpha$ and $|a_{i,f} - a_{i,g}|_{U(s)} < \alpha$ for all i .

Remark. When $f = g$, the existence and uniqueness of the factorization given in equation (4.6) is the standard Weierstrass Preparation Theorem, and the proof of Lemma 1 will be a modification of the standard proof (see [7]).

Proof. Choose $0 < \rho < r_n$. Since $f(0; w) = aw^N + \dots$, there exists $t = (t_1, \dots, t_{n-1}) > 0$ such that for $z \in U(t)$ and for $|w| = \rho$ we have $|f(z; w)| > |a|\rho^N/2$. Let $\sigma = |a|\rho^N/4$ and choose $\epsilon < \sigma$. If $|f - g|_{U(t)} < \epsilon$, then for $z \in U(t)$ and $|w| = \rho$ we have $|g(z; w)| > \sigma$ and $|f(z; w) - g(z; w)| < |f(z; w)|$. Thus, for $z \in U(t)$, the equation $f(z; w) = 0$ each has exactly N roots in the disk $|w| < \rho$ (by the winding number theorem) and the equation $g(z; w) = 0$ each has exactly N roots in the disk $|w| < \rho$ (by Rouché's theorem). Let w_1, \dots, w_N be the roots of $g(z; w) = 0$ for a fixed $z \in U(t)$. Then, for every $\nu > 0$ we have

$$(4.7) \quad b_{\nu,g}(z) = \sum_{i=1}^N w_i^\nu = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{w^\nu (\partial g / \partial w)(z; w)}{g(z; w)} dw.$$

Thus the $b_{\nu,g}$ are holomorphic functions of z for $z \in U(t)$. Now let $\sigma_{0,g}(z), \dots, \sigma_{N,g}(z)$ be the elementary symmetric polynomials in w_1, \dots, w_N . Then the $\sigma_{i,g}$ are polynomials in the $b_{\nu,g}$ and the coefficients of the polynomials are rational numbers, independent of z . Thus the function

$$p_g(z; w) = \sum_{i=0}^N (-1)^i \sigma_{i,g}(z) w^{N-i} = \sum_{i=0}^N a_{i,g}(z) w^{N-i}$$

is holomorphic on $U(s)$ where $s = (t, \rho)$, and for a fixed $z \in U(t)$, it vanishes on exactly the same set as $g(z; w) = 0$.

Let $u(z; w) = g(z; w)/p(z; w)$. Then u is meromorphic on $U(s)$ and for a fixed z , h has only removable singularities in the variable w . Thus u can be extended to a nowhere vanishing function on all of $U(s)$ which is analytic in w for each fixed z and which is also analytic on the complement of the set where $p(z; w) = 0$. In particular, $u(z; \xi)$ is holomorphic in an open neighborhood of the set $\Gamma = U(t) \times \{\xi : |\xi| = \rho\}$. Thus, if we write

$$u(z; w) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{u(z; \xi)}{\xi - w} d\xi$$

we see that $u(z; w)$ is holomorphic on $U(s)$.

To prove that p_g , u_g and $a_{i,g}$ are continuous at f , it suffices to prove for each ν , that $b_{\nu,g}$ is continuous at f (since the $a_{i,g}$ are polynomials in the $b_{\nu,g}$).

Again we shall use the integral expression in equation (4.7): The formula for $b_{\nu,f}$ is obtained from (4.7) by replacing g with f . To show $b_{\nu,g}$ is close to $b_{\nu,f}$ when $|g - f|_{U(r)}$ is small, it suffices to show that the corresponding integrands are uniformly close, that is, we must show that

$$\frac{(\partial g / \partial w)(z; w)}{g(z; w)} \quad \text{is close to} \quad \frac{(\partial f / \partial w)(z; w)}{f(z; w)}$$

is small when $(z; w) \in \Gamma$. By assumption, the denominators are close. They are also bounded away from zero since, by the choice of ϵ , we have $|f(z; w)| > \sigma$ and $|g(z; w)| > \sigma$ when $(z; w) \in \Gamma$. Since the denominators are close and bounded away from zero, it suffices to show that the numerators are close. But this follows easily from the Cauchy Integral Formula:

$$\frac{\partial(g - f)}{\partial w}(z; w) = \frac{1}{2\pi i} \int_{|w|=r} \frac{(g - f)(z; \xi)}{(\xi - w)^2} d\xi,$$

which shows that the left-hand side is bounded above by $\epsilon r / (r - \rho)^2$.

It remains to prove that u_g are continuous at $g = f$: We have the estimate:

$$|u_g(z; w)^{-1} - u_f(z; w)^{-1}| = |p_f g - p_g f| / |f g| < |p_f g - p_g f| \sigma^{-2}$$

if $z \in U(t)$ and $|w| = \rho$. Thus, if g is close to f then, as we have already proved, p_g is close to p_f , and thus $|u_g(z; w)^{-1} - u_f(z; w)^{-1}|$ is small for $z \in U(t)$ and $|w| = \rho$. Hence, by the maximum modulus principle, it is small for $|w| < \rho$ as well. From this we deduce that $|u_f / u_g - 1| = |u_f| |u_g^{-1} - u_f^{-1}|$ is small, and hence, so is $|u_g / u_f - 1|$ and $|u_f - u_g| = |u_f| |u_g / u_f - 1|$.

LEMMA 2. Fix $N > 0$. Then there exist $P_0, P_1, \dots, P_N \in \mathbf{Z}[A_{-1}, A_{-2}, \dots][[A_1, \dots, A_N]]$ with the following properties: Let $g(z; w)$ be holomorphic on $U(r) \subseteq \mathbf{C}^n$ and assume N is the order of vanishing of $g(0; w)$ at the origin. Write

$$g(z; w) = \sum_{i=0}^{\infty} a_{N-i}(z) w^i$$

and assume $a_0(z) = 1$. Let $p(z; w)$ be the corresponding Weierstrass polynomial:

$$p(z; w) = \sum_{i=0}^N p_{N-i}(z) w^i.$$

Then for all j , $p_j(z) = P_j(\{a_i(z)\})$ (here i ranges over all nonzero integers less than or equal to N).

Proof. We choose $\rho > 0$ and $t = (t_1, \dots, t_{n-1}) > 0$ such that $|g(z; w) - w^N| < |w^N|$ for $z \in U(t)$ and $|w| = \rho$. Then

$$\tilde{g}(z; w) = w^{-N} g(z; w) - 1 = \sum_{\substack{N \geq i > -\infty \\ i \neq 0}} \frac{a_i}{w^i}$$

has the property: $|\tilde{g}(z; w)| < 1$ for $z \in U(t)$ and $|w| = \rho$. Formula (4.7) tells us

$$(4.8) \quad b_{\nu, g}(z) = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(\partial g / \partial w)(z; w)}{w^{N-\nu}(1 + \tilde{g}(z; w))} dw$$

for all $\nu > 0$. Now we simply expand $1/(1 + \tilde{g}(z; w))$ as a geometric series: The coefficient of $w^{N-\nu-1}$ is easily seen to be an element of $\mathbf{Z}[A_{-1}, A_{-2}, \dots][[A_1, \dots, A_N]]$ evaluated on the a_i . This, together with (4.8) and the fact that the $a_i(z)$ are polynomials in the $b_\nu(z)$ yields the result of Lemma 2.

Step 2. Reduction to the case of Weierstrass Polynomials. Next we shall show how Lemma 1 allows us to reduce, in the proofs of Theorem 6, 7 and 8, to the case where f and g are W-Polynomials (Weierstrass polynomials, that is, polynomials in one variable, with coefficients which are analytic in the other variables) of degree N .

Fix f as in Theorems 6, 7 and 8. We use the following notation: If ϕ_1 and ϕ_2 are functions defined on a compact set K , then we say $\phi_1 \underset{K}{\sim} \phi_2$ (or simply $\phi_1 \sim \phi_2$ if there is no possibility of confusion) if there is a positive constant $c = c(f)$, depending only on f , such that $c^{-1}\phi_1(x) \leq \phi_2(x) \leq c\phi_1(x)$ for all $x \in K$.

In order to apply the Weierstrass Preparation Theorem, we need to fix a choice of coordinate system. In this section, we shall choose a *generic* system of coordinates: By this, we mean that if N is the order of vanishing of $f(z; w)$ at the origin, then $f(0; w)$ also vanishes to order N at the origin. To see that such a coordinate system exists, write $f = \sum_{i=N}^{\infty} f_i$ where $f_i(x_1, \dots, x_n)$ is a homogeneous polynomial of degree i and $f_N \neq 0$. Since $f_N \neq 0$, there exists $v \in \mathbf{R}^n$ such that $f_N(v) \neq 0$. Now choose a vector space basis v_1, v_2, \dots, v_n of \mathbf{R}^n in such a way that $v_n = v$, and let $\tilde{f} = \sum_{i=N}^{\infty} \tilde{f}_i$ denote the function f in the new coordinate system, that is $\tilde{f}(z_1, z_2, \dots, z_{n-1}; w) = f(z_1 v_1 + \dots + z_{n-1} v_{n-1} + w v)$. Then $\tilde{f}_N(0; w) = f_N(w v) = f_N(v) w^N \neq 0$ and hence we see that $\tilde{f}(0; w)$ vanishes to order N at the origin.

First we make the reduction to the case of W-Polynomials for Theorem 6: Lemma 1 implies that f has a Weierstrass factorization $f = u_f p_f$ on $I(r')$ for some $r' < r$. Moreover, for every $\epsilon' > 0$ there exists $\epsilon > 0$ such that

$$(4.9) \quad |f - g|_{I(r)} < \epsilon \quad \text{implies} \quad |p_f - p_g|_{I(r')} < \epsilon'.$$

Since u_f is nonvanishing on $I(r')$ we deduce it is bounded above and below by

positive constants, depending only on f : thus we have $f \underset{I(r')}{\sim} p_f$. Assume that f satisfies the hypothesis of Theorem 6: $\int_{I(r)} |f|^{-\delta} < \infty$. Then we have

$$\int_{I(r')} |p_f|^{-\delta} < \infty .$$

Now assume that Theorem 6 has been proved for W-Polynomials of degree N . Applying the conclusion of Theorem 6 to p_f , we deduce that there is $0 < s < r'$ and an $\epsilon' > 0$ such that

$$(4.10) \quad \sup_p \left\{ \int_{I(s)} |p|^{-\delta} : |p_f - p|_{I(r')} < \epsilon', p \text{ W-Polynomial of degree } N \right\} < \infty .$$

Choose $\epsilon > 0$ so that condition (4.9) holds and such that $1/2 < |u_g/u_f|_{I(r')} < 2$. Combining (4.9) and (4.10) we obtain:

$$\sup_g \left\{ \int_{I(s)} |p_g|^{-\delta} : |f - g|_{I(r)} < \epsilon \right\} < \infty$$

Finally, the estimate $1/2 < |u_g/u_f|_{I(r')} < 2$ guarantees that $g \underset{I(s)}{\sim} p_g$. Hence, if g ranges over the set of holomorphic functions on $U(r)$, satisfying $|f - g|_{I(r)} < \epsilon$, we have $\sup_g \int_{I(s)} |g|^{-\delta} < \infty$. This completes the reduction step for Theorem 6. The reduction steps for Theorem 7 are similar.

The reduction step for Theorem 8 is as follows: The proof of Part A does not require a reduction to the Weierstrass polynomial case, and thus we concentrate on the reductions of Parts B, C, and D. Assume that Theorem 8 D has been established for the class of W-Polynomials of a fixed degree N . Let f satisfy the hypothesis in D of Theorem 8, i.e., f is not a local N power. Then, by Lemma 1, there exists a $0 < s < r$ such that f has a Weierstrass factorization on $I(s)$: $f = u_f p_f$. Moreover, since u_f is nowhere vanishing, it is an N^{th} power of a real analytic function on $I(s)$. Thus f is an N^{th} power of a real analytic function on $I(s)$ if and only if p_f is. Since Theorem 7 D is true for p_f , we have $\int_{I(s)} |p_f|^{-\delta} < \infty$ for some $\delta > 1/N$. But u_f is nowhere vanishing and is thus bounded above and below by positive constants. This implies that $f \underset{I(s)}{\sim} p_f$. Hence, $\int_{I(s)} |f|^{-\delta} < \infty$. Now we must show that there exists $\epsilon > 0$ such that (4.4) holds, but this argument is identical to that used in the reduction step of Theorem 6, and so we omit it. The reduction of Part B of Theorem 8 to a Weierstrass polynomial is similar.

Now assume Theorem 8 C has been established for W-Polynomials. Then, as above, $f \underset{I(s)}{\sim} p_f$ so $\int_{I(s)} |f|^{-\delta} = \infty$ if and only if $\int_{I(s)} |p_f|^{-\delta} = \infty$. Again, we can apply Theorem 8 C for W-polynomials since p_f is an N^{th} power if f is. Finally we observe that if a W-Polynomial is the N^{th} power of a real analytic function on $I(s)$, then its N^{th} root is another W-Polynomial. The remainder of the reduction step for Theorem 8 C is identical to that for Theorem 8 D, and so we shall omit it.

We now are in position to prove Theorems 6, 7 and 8. From the discussion above, we may assume that $f = p$, a Weierstrass polynomial. Let $p(x; y) = p(x_1, x_2, \dots, x_{n-1}; y) = \sum a_i(x)y^{N-i}$ be a Weierstrass polynomial. Write $I(r) = I^{(n-1)}(r_1, \dots, r_{n-1}) \times I^{(1)}(r_n)$ so that

$$\int_{I(r)} |p(x; y)|^{-\delta} dx dy = \int_{I^{(n-1)}(r_1, \dots, r_{n-1})} \left[\int_{I^{(1)}(r_n)} |p(x; y)|^{-\delta} dy \right] dx.$$

For each fixed value of x , the inner integral is of the form: $\int_I |P(y)|^{-\delta} dy$, where P is a polynomial of one variable, and $I \subseteq \mathbf{R}$ is an interval. Thus our one-dimensional estimates apply.

Step 3. Proof of Theorem 8. We start with the proof of part A: Let $f = \sum_{i \geq N} f_i$ with f_i homogeneous of degree i and $f_N \neq 0$. For $\rho > 0$ we let

$$B(\rho) = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1^2 + \dots + x_n^2 = \rho^2\} \quad \text{and} \quad S^n = \left\{ \mu \in \mathbf{R}^n : \sum \mu_i^2 = 1 \right\}.$$

Choose ρ so that $B(\rho) \subseteq I(s)$ and let $d\mu$ be the standard euclidean measure on the unit sphere in \mathbf{R}^n . Then

$$\int_{B(\rho)} |f(x_1, \dots, x_n)|^{-\delta} dx_1 \dots dx_n = \int_{S^n} \int_0^\rho |f(r\mu_1, \dots, r\mu_n)|^{-\delta} r^{n-1} dr d\mu.$$

Now for r small, we have $f(r\mu_1, \dots, r\mu_n) \sim r^N f_N(\mu_1, \dots, \mu_n)$ provided $f_N(\mu_1, \dots, \mu_n) \neq 0$. Restricting μ to a region in S^n where $f_N(\mu_1, \dots, \mu_n)$ is bounded below by a positive constant, we see that

$$\int_{B(\rho)} |f(x_1, \dots, x_n)|^{-\delta} dx_1 \dots dx_n \geq C \int_0^\rho r^{-N\delta+n-1} dr,$$

which is infinite when $\delta \geq n/N$. This completes the proof of part A.

Now we prove part C: By Step 2, we may assume that $f(x; y) = p(x; y) = \sum_{j=0}^N a_j(x)y^{N-j}$ where p is a Weierstrass polynomial of degree N in the variable y (here we are using the notation $x = (x_1, \dots, x_{n-1})$ and $y = x_n$). We wish to estimate:

$$\mathcal{I}(p) = \int_{I(r)} |p(x; y)|^{-\delta} dx dy = \int_{I^{(n-1)}(r_1, \dots, r_{n-1})} Q_I(x) dx,$$

where $I = I^{(1)}(r_n)$.

We first assume that $p = q^N$ where q is a polynomial of degree one in y : $q(x; y) = y - a(x)$. Here a is real analytic, and by assumption, $a(0) = 0$. Then $Q_I(x) = \int_I |y - a(x)|^{-\delta N} dy$. Thus, if $\delta \geq 1/N$, we see that $Q_I(x)$ is infinite for all x sufficiently small, which implies that $\int_{I(r)} |p(x; y)|^{-\delta} dx dy = \infty$.

To prove part B, we suppose that $\delta < 1/N$ and let $p(x, y)$ be a W-Polynomial of degree N . Then the estimate (1.9) in Theorem 2 implies that the inner integral, $Q_I(x)$, is bounded by $Cr_n^{1-N\delta}$ where $C = C(\delta, N)$ is a constant which depends only on N . Thus

$$\int_{I(r)} |p(x; y)|^{-\delta} dx dy < 2^n C(\delta, N) r_1 r_2 \cdots r_n^{1-N\delta} .$$

Since this bound is independent of p , we see that (4.3) holds (with any choice of $\epsilon > 0$ and $0 < s < r$). This completes the proof of part B.

Finally we prove part D: Assume that $p(x; y)$ is not an N^{th} power. Choose δ such that $1/N < \delta < 1/(N - 1)$. Then, by (1.10) in Theorem 2, we know that $Q_I(x) \sim 1/L_0(x)^{N\delta-1}$. In view of the relation (2.10) between scale clusters and generalized discriminants, $L_0(x) \sim \Delta_1(x)$, so $Q_I \sim 1/\Delta_1^{N\delta-1}$ (note that by the first remark following Theorem 7, we may assume that p is real valued, and hence, (2.10) does apply). By (2.14), $\Delta_1(x) \sim \sum_{m=1}^M |U_m(x)|^{1/m}$. We note that the $U_m(x)$ are real analytic on $I^{(n-1)}(r)$: This follows from the fact that the $U_m(x)$ are polynomials in the $a_j(x)$ which, by assumption, are real analytic functions. Now we must have $U_m \neq 0$ for some m : Otherwise, $\Delta_1 = 0$ which implies $L_0 = 0$, which implies that for every x , all N roots of $p(x; y) = 0$ are equal, contradicting the assumption that p is not an N^{th} power. Choose m so that $U_m \neq 0$. Then we have

$$\int_{I^{(n)}} |p(x; y)|^{-\delta} dx dy < C \int_{I^{(n-1)}} |U_m(x)|^{-\delta'} dx$$

where $\delta' = (N\delta - 1)/m$ and $C = C(\delta, N) > 0$. Now, making a linear change of variables if necessary, we may assume that $U_m(0, 0, \dots, 0, x_{n-1}) \neq 0$. Let N' be the order of vanishing of $U_m(0, 0, \dots, 0, x_{n-1})$ at the origin. Choosing $\delta > 1/N$ so that $\delta' < 1/N'$ (i.e., choose δ so that $1/N < \delta < 1/N + m/NN'$), we see that $\int_{I(r)} |p(x; y)|^{-\delta} dx dy$ is finite.

To complete the proof of part D, we must show the existence of $\epsilon > 0$ for which (4.4) holds. We fix δ (and hence $\delta' = (N\delta - 1)/m$) as above. Then if $g(x; y) = q(x; y)$ is a Weierstrass polynomial of degree N we have

$$(4.11) \quad \int_{I^{(n)}(s)} |q(x; y)|^{-\delta} dx dy < C(\delta, N) \int_{I^{(n-1)}(s)} |U_{m,q}(x)|^{-\delta'} dx$$

for all $0 < s < r$.

By Lemma 1, there exists $0 < s' < r$ such that if $q(x; y)$ is close to $p(x; y)$ on $U^{(n)}(r)$, then $a_{i,p}(x)$ is close to $a_{i,q}(x)$ on $U^{(n-1)}(s')$ and hence $U_{m,q}(x)$ is close to $U_{m,p}(x)$ on $U^{(n-1)}(s')$. By Theorem 8 B, there exists $0 < s < s'$ such that if $U_{m,q}(x)$ is close to $U_{m,p}(x)$ on $U^{(n-1)}(s')$, then $\int_{I^{(n-1)}(s)} |U_{m,q}(x)|^{-\delta'} dx$ is bounded above by a constant which depends only on p . Hence the left-hand side of (4.11)

is bounded above by a constant which depends only on p , and this completes the proof of Theorem 8.

We now provide a justification for Remark 2: To do this, we use a slightly simpler version of U_m : As we observed in the proof of part D, we have $Q_I(x) \sim 1/L_0(x)^{N\delta-1}$ and

$$\int_{I(r)} |f(x; y)|^{-\delta} dx dy \sim \int_{I(r)} |p(x; y)|^{-\delta} dx dy \sim \int_{I^{(n-1)}} |L_0(x)|^{-(N\delta-1)} dx.$$

Now one easily sees that

$$\begin{aligned} L_0(x) &= L_0(\{w : p(x; w) = 0\}) \\ &= L_0\{w : p(x, w - \frac{P_1}{N}) = 0\} \\ &\sim \sup \left\{ w : p\left(x, w - \frac{P_1}{N}\right) = 0 \right\}. \end{aligned}$$

Hence, if we write

$$p\left(z; w - \frac{P_1}{N}\right) = u_N(z) + u_{N-1}(z)w + \cdots + u_2(z)w^{N-2} + w^N,$$

we see that

$$\int_{I(r)} |f(x; y)|^{-\delta} dx dy \sim \sum_{m=2}^N \int_{I^{(n-1)}} |u_m(x)|^{-\delta'} dx.$$

This establishes part 3 of Remark 2. The rest follows from Lemma 2.

Step 4. Proof of Theorem 6. We shall need the following one variable stability result:

LEMMA 3. *If $f_1(y), \dots, f_l(y)$ are real analytic on $I(r)$ and if $\delta_1, \delta_2, \dots, \delta_l$ are positive numbers such that*

$$(4.12) \quad \int_{I(r)} \prod_{i=1}^l |f_i(y)|^{-\delta_i} dy < \infty$$

then there exists $\epsilon > 0$ and $0 < s < r$ such that

$$(4.13) \quad \sup_{g_i} \left\{ \int_{I(s)} \prod_{i=1}^l |g_i(y)|^{-\delta_i} dy : g_i \text{ holomorphic on } U(r), |g_i - f_i|_{U(r)} < \epsilon \right\} < \infty.$$

Proof. After shrinking r , if necessary, we may assume that $f_i(y) \neq 0$ if $y \in U(r)$ and $y \neq 0$. Lemma 1 (applied in the case $n = 1$) shows, as in the previous reduction arguments, that we may assume that the f_i and the g_i are W-Polynomials. This means that they are ordinary monic polynomials of one variable with constant coefficients. Since all the zeros of the f_i are at the origin, we see that $f_i = y^{N_i}$ for some $N_i \geq 0$, and that g_i is a monic polynomials of degree N_i for all i . Fix $0 < s < r$ as in Lemma 1. Then, by choosing ϵ small enough, we can ensure that $I(s)$ will be g_i -admissible for all i . From (4.12) we conclude that $\sum N_i \delta_i < 1$. Thus we can apply part (b) of Claim 1 to get the desired bound.

We return to the proof of Theorem 6. We know, by Step 2, that we can assume that $f = p$, a Weierstrass polynomial. We observe that we can choose p in such a way that it has degree N , where N is the order of vanishing of f : Choose a coordinate system for f such that the following holds: If $f(x, y) = \sum a_{i,j} x^i y^j$ and if $a_{i,j} \neq 0$ for some (i, j) , then $a_{\mu,\nu} \neq 0$ for all (μ, ν) such that $\mu + \nu = i + j$. If we apply Step 2 to f in this coordinate system, then we reduce to the case that $f = p$, with p of degree N . By Theorem 7A, we may assume that $\delta' < 2/N$. Again by Theorem 7, we may assume that $1/N \leq \delta'$. Replacing p by p^2 , δ by $\delta/2$, and N by $2N$, we may assume that N is even. Now an important property of the integral $\int_{I(r)} |f|^{-s}$ is that it will remain finite for slightly larger values of s if it is finite for $s = \delta$. This is established in detail in [21] (when an asymptotic expansion of the form (3.22) is available for the distribution function $F(\epsilon)$, this also follows from (3.23)). Thus we need to establish Theorem 2 only for a dense set of δ . We may assume then that $1/(N - k) < \delta' < 1/(N - k - 1)$ for some k such that $0 \leq k \leq (N/2) - 1$.

Now fix a k such that $\frac{1}{N-k} < \delta' < \frac{1}{N-k-1}$. We have, according to (b) of Theorem 2,

$$\iint_{I(r)} |p(x, y)|^{-\delta} dx dy = \int_{I(v)} Q_I(p)(x) dx$$

and

$$Q_I(p)(x) \sim \Delta_k(p)(x)^{-a} \Delta_{k+1}(p)(x)^{-b},$$

where $a = (N - k)\delta - 1$ and $b = 1 - (N - k - 1)\delta$. Now, by (1.12) of Theorem 2, we have $\Delta_k(p) \sim \sum_{m=1}^M |U_m^{(k)}(p)|^{1/m}$. Let $\nu = \nu(m, k)$ be the order of vanishing of $U_m^{(k)}(p)(x)$ and let m_0 be a value for m for which $\nu(m, k)/m$ is minimized. Similarly, let n_0 be a value for n for which $\nu(n, k + 1)/n$ is minimized. Then, near the origin, we have

$$Q_I(p)(x) \sim |U_{m_0}^{(k)}(p)(x)|^{-a/m_0} |U_{n_0}^{(k+1)}(p)(x)|^{-b/n_0}.$$

Now if q is a Weierstrass polynomial which is close to p , then, by Lemma 1, $a_{i,p}$

is close to $a_{i,q}$. Thus, again by (c) of Theorem 2, $U_m(p)$ is close to $U_m(q)$ and so we can apply Lemma 3 to obtain the desired result.

Step 5. Proof of Theorem 7. As in the proof of Theorem 6, we reduce to the case where $f = p$, a Weierstrass polynomial of degree N , where N is the multiplicity of f at the origin and N is even. Also, as before, we need only establish the result for a dense set of δ in the interval $1/N < \delta < 2/N$. In particular, we can assume that δ is a rational number, and apply Theorem 2 part (b) to conclude:

$$(4.14) \quad \iiint_{I(r)} |p(x, y, z)|^{-\delta} dx dy dz \sim \iint_{I^{(2)}(r)} |\phi(x, y)|^{-\delta'} dx dy$$

where $\phi(x, y) = \Phi(a_1(x, y), \dots, a_N(x, y))$. Now by Lemma 1, a small deformation of f will produce small perturbations of the $a_i(x, y)$ and thus, a small perturbation of $\phi(x, y)$. Thus we can use (4.14) to reduce Theorem 7 to Theorem 6, which has already been proved.

V. Discussion and open questions. In this section, we discuss briefly some related results in the literature, as well as possibilities for further developments.

(1) In this paper, we have focused on the stability of integrals of the form (1.2). It is likely that our methods can be suitably adapted to handle the closely related oscillatory integral case (1.1). A natural immediate goal would be a new proof of Karpushkin's theorem in two dimensions, as well as a version of our Theorems 7 and 8 in all dimensions.

(2) Varchenko's counterexample shows that we may not have stability for δ arbitrarily close to δ_0 , where δ_0 is the supremum of all δ for which the integral (1.2) is finite. But it suggests, and this is confirmed by Theorem 7 and 8 in the context of the integrals (1.2), that ranges of δ exist in which stability would hold. As we also saw in Theorems 7 and 8, these ranges seem to be closely related to the natural restriction $0 < \delta < \frac{2}{N}$ of Theorem 2. Indeed, an argument along the lines of Theorem 6–8 would require a version of Theorem 2 in the range $0 < \delta < \frac{d}{N}$. This line of investigation may deserve further effort.

(3) For functions of one variable, the notion of multiplicity of a function f at a point is unique, and can be defined equivalently as either δ_0^{-1} or its order of vanishing at the point. This is no longer the case in higher dimensions, which explains perhaps the subtleties inherent to stability questions. Besides δ_0 , natural generalizations of multiplicity include N which we have relied on in this paper, and the more geometric notion of Milnor number $\mu(f)$ [12], which is the winding number of $|\nabla f|/|\nabla f|$ over a small sphere around the origin. This last notion is also the one arising naturally in the study of Radon transforms [14]. It would be very attractive if the ranges of stability for both types of integrals (1.1) and (1.2)

can be formulated in terms of the Milnor number of f , for example in the ranges needed for the regularity of Radon transforms in higher codimension.

(4) Similar questions of range and stability arise for estimates of the form (a) in Theorem 4 for the growth of distribution functions. In this context, we note the recent results of A. Carbery, M. Christ, and J. Wright (to appear) which provide uniform estimates for the distribution function in terms of individual vertices in the Newton diagram.

(5) The case of stability of the integrals (1.2) for holomorphic functions is of considerable interest. Conversely, it may be interesting to try to extend our methods to classes of smooth functions satisfying some finite multiplicity conditions.

(6) Another approach to stability would be to relax the requirement of uniformity of the estimates (1.1) and (1.2) with respect to *all* g close to f . An early result of Lichtin [11] shows that, in the holomorphic set-up and for one-parameter deformations, the order of the leading pole in (1.1) is semi-continuous. What would be particularly useful in, e.g., the study of complex Monge-Ampere equations, is uniformity and convergence for a sequence of deformations of f . In particular, Tian [22] has shown that if f_n are holomorphic functions on the unit disk in \mathbf{C}^2 , converging uniformly to a function f with the integral (1.2) over the unit disk finite, then the corresponding integrals of $|f_n|^{-\delta'}$ over the disk of radius $1/2$ converge to the integral of $|f|^{-\delta'}$ over the same disk, for any $\delta' < \delta$. In higher dimensions, using an extension theorem of Ohsawa-Takegoshi [13], Siu [19, 20] has shown that if $f(z_1, \dots, z_n, w)$ is any holomorphic function on the polydisk in \mathbf{C}^{n+1} with $|f(z, 0)|^{-\delta}$ integrable in a small disk near the origin, then there is a sequence $w_n \rightarrow 0$ for which the integrals of $|f(z, w_n)|^{-\delta}$ over a smaller disk would converge to the integral of $|f(z, 0)|^{-\delta}$ over the same smaller disk.

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