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# Multilinear level set operators, oscillatory integral operators, and Newton polyhedra*

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## 1. Introduction

There is increasing evidence that many central questions in analysis may depend on the stability of certain estimates under perturbations. Typically, to a function $S(x)$ are associated many invariants, such as the growth rate of its distribution function, the decay rates of the oscillatory integrals or oscillatory integral operators with $S(x)$ as phase, and the sharp constants entering such estimates. The issue of stability can take many forms:

- Given $S(x)$, the issue is to determine the behavior of the above invariants for all functions $\tilde{S}(x)$ in a sufficiently small neighborhood of $S(x)$, with respect to a suitable topology. Under this form, stability has been investigated by Karpushkin [5], Tian [18], Siu [14–15], and by the authors in [10] and [12];
- More globally, we may also seek uniform bounds for the above invariants which hold for *all* $\tilde{S}(x)$, subject to, say, some given uniform lower bounds on the derivatives of $\tilde{S}(x)$. In this formulation, stability has been investigated by Carbery, Christ, and Wright [2].

In this paper, we shall investigate the latter form of stability for certain multilinear operators defined by $S(x)$ and focus on the emergence of the Newton polyhedron of $S$ in this context.

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More precisely, let $U = \{(x_1, x_2, ..., x_d) \in \mathbf{R}^d : 0 < x_i < 1\}$ and let $D$ be a subset of $U$. We shall always assume that $d \geq 2$. We shall consider two classes of multilinear operators. The first is the *multilinear oscillatory integral operator*

$$T_D(f_1, ..., f_d) = \int_D e^{i\lambda S(x_1,...,x_d)} f_1(x_1) \cdots f_d(x_d)\, dx_1 \cdots dx_d \;, \qquad (1.1)$$

defined by a polynomial phase function $S(x_1, ..., x_d) \in \mathbf{R}[x_1, ..., x_d]$, and the second the *multilinear level set operator*

$$W_D(f_1, ..., f_d) \;=\; \int_D \chi_M(x_1, ..., x_d) f_1(x_1) \cdots f_d(x_d)\, dx_1 \cdots dx_d, \qquad (1.2)$$

where $F(x_1, \cdots, x_d) \in \mathbf{R}[x_1, \cdots, x_d]$ is also a polynomial and $\chi_M$ is the characteristic function of $\{x : |F(x)| \leq M\}$. These operators are natural generalizations of the familiar oscillatory and level set operators studied in [6–9,13] and [2]. We shall see shortly that multilinearity is the proper analytic notion associated with the geometric notion of Newton polyhedron.

To formulate our results, we need some preliminary concepts. First, recall that the norm of a multi-linear operator $T(f_1, \cdots, f_d)$ on the space $\otimes_{i=1}^d L^{p_i}[0, 1]$ is defined by

$$||T|| \;= \sup |T(f_1, ..., f_d)|,$$

where the sup is taken over all $f_i$ such that $||f_i||_{p_i} \leq 1$. Next, we introduce the key concept of *algebraic domain* as follows. Let $r, n, d$ be positive integers. Then $D \subseteq U$ is called a simple algebraic domain of type $(r, n, d)$ if $D$ is of the form:

$$D = \{x = (x_1, ..., x_d) \in U : \; f_k(x) \geq A_k, \;\; k = 1, 2, ..., r'\} \qquad (1.3)$$

where the $f_k \in \mathbf{R}[x]$ are non-constant polynomials of degree at most $n$, $r' \leq r$, $A_k \in \mathbf{R}$. We say that $D$ is an algebraic domain of type $(r, n, d, w)$ if $D = D_1 \cup \cdots \cup D_{w'}$ where the $D_i$ are simple algebraic domains of type $r, n, d$, and $w' \leq w$.

The main results of this paper are the following. The notation is the standard one for partial derivatives on $\mathbf{R}^d$:

$$F^{(\alpha)} = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \;\; \alpha = (\alpha_1, \cdots, \alpha_d), \;\; |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

A multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ will often be viewed as a point in $\mathbf{R}^d_{\geq 0}$, and referred to as a vertex. Given $\alpha \in \mathbf{R}^d_{\geq 0}$, define $P(\alpha) = \{\alpha + x : x = (x_1, ..., x_d) \in \mathbf{R}^d_{\geq 0}, x_i \geq 0, \; 1 \leq i \leq d\}$. Let $\alpha^{(1)}, ..., \alpha^{(K)} \in \mathbf{R}^d_{\geq 0}$ be a collection of vertices with non-negative entries. Then the Newton polyhedron $N(\alpha^{(1)}, ..., \alpha^{(K)})$ generated by $\alpha^{(1)}, ..., \alpha^{(K)}$ is defined to be the convex hull of the set $\cup_{k=1}^K P(\alpha^{(k)})$.

**Theorem A.** (Sublevel Set Estimates) *Let* $\alpha^{(1)}, ..., \alpha^{(K)} \in \mathbf{N}^d \setminus \{0\}$ *be* $K$ *given vertices, and let* $F \in \mathbf{R}[x_1, ..., x_d]$ *be any polynomial of degree* $n_F$. *Set*

$$D(\alpha^{(1)}, \cdots, \alpha^{(K)}) = \left\{x \in U; \quad |F^{(\alpha^{(k)})}(x)| > 1, \quad 1 \le k \le K\right\}$$

*Then for any domain* $D \subseteq D(\alpha^{(1)}, \cdots, \alpha^{(K)})$ *and any* $\alpha \in N(\alpha^{(1)}, ..., \alpha^{(K)})$, *we have*

$$|W_D(f_1, ..., f_d)| \le C M^{\frac{1}{|\alpha|}} \ln^{d-2}\left(2 + \frac{1}{M}\right) \prod_{i=1}^{d} ||f_i||_{p_i}, \quad d \ge 2. \tag{1.4}$$

*Here* $M$ *is any positive number, and* $\frac{1}{q_i} = 1 - \frac{1}{p_i} = \frac{\alpha_i}{|\alpha|}$. *The constant* $C$ *depends only on* $n_F$ *and* $\alpha^{(1)}, \cdots, \alpha^{(K)}$.

**Theorem B.** (Oscillatory Integral Estimates) *Let* $\alpha^{(1)}, ..., \alpha^{(K)} \in \mathbf{N}^d \setminus \{0\}$ *be* $K$ *given vertices, and let* $S \in \mathbf{R}[x_1, ..., x_d]$ *be any polynomial of degree* $n_S$. *Set*

$$D(\alpha^{(1)}, \cdots, \alpha^{(K)}) = \left\{x \in U; \quad |S^{(\alpha^{(k)})}(x)| > 1, \quad 1 \le k \le K\right\}$$

*Let* $N^*(\alpha^{(1)}, \cdots, \alpha^{(K)})$ *be the reduced Newton polyhedron generated by the vertices* $\alpha^{(k)}$, *i.e., the Newton polyhedron generated only by those vertices* $\alpha^{(k)}$ *with at least two strictly positive components. Then for any algebraic domain* $D \subseteq D(\alpha^{(1)}, \cdots, \alpha^{(K)})$ *and any* $\alpha \in N^*(\alpha^{(1)}, ..., \alpha^{(K)})$, *we have*

$$|T_D(f_1, ..., f_d)| \le C|\lambda|^{-\frac{1}{|\alpha|}} \ln^{d-\frac{1}{2}}(2 + |\lambda|) \prod_{i=1}^{d} ||f_i||_{p_i}, \quad d \ge 2. \tag{1.5}$$

*Here* $\lambda$ *is any number, and* $\frac{1}{q_i} = 1 - \frac{1}{p_i} = \frac{\alpha_i}{|\alpha|}$. *The constant* $C$ *depends only on* $n_S$, $|\alpha|$, *and the type* $r, n, d, w$ *of* $D$.

One particularly attractive feature of the estimates (1.4–1.5) may be worth stressing: to each point $\alpha$ in the Newton polyhedron $N(\alpha^{(1)}, \cdots, \alpha^{(K)})$ (or $N^*(\alpha^{(1)}, \cdots, \alpha^{(K)})$), corresponds an estimate with $L^p$-spaces determined by the "slopes" $\frac{\alpha_i}{|\alpha|}$, $1 \le i \le d$, and decay rate determined by the "distance" $|\alpha|$. The $p_i$ and $q_i$ satisfy the constraint

$$\sum_{i=1}^{d} \frac{1}{q_i} = 1, \qquad \sum_{i=1}^{d} \frac{1}{p_i} = d - 1. \tag{1.6}$$

Remarkably, a key example of such an estimate had been obtained earlier by Carbery, Christ, and Wright [2], namely in the context of Theorem A, for a single vertex in dimension $d = 2$ and when $D$ is the full unit square $U$. This estimate also plays an important role in our approach. However, given its importance, and given the fact that we actually need a generalization of this estimate to the case of a general algebraic domain $D$, we have presented an independent treatment. Our treatment applies only in the present setting of polynomial functions $F(x)$ and $S(x)$, but not to the more general setting of [2] (c.f. Section II, Steps 2 and 3).

Evidently, the best estimates are obtained for $\alpha$ on the boundary of the Newton polyhedron. If we choose $\alpha = (\alpha_1, \cdots, \alpha_d)$ to be the point on the boundary of the Newton polyhedron which satisfies the condition $\alpha_1 = \cdots = \alpha_d$, then all the $L^p$-spaces are $L^{\frac{d}{d-1}}$, and the estimates (1.4) and (1.5) become respectively

$$\begin{aligned} ||W_D(f_1, \cdots, f_d)|| &\leq C\, M^{\frac{1}{d}\delta} \ln^{d-2}\left(2 + \frac{1}{M}\right) \prod_{i=1}^{d} ||f_i||_{\frac{d}{d-1}} \\ ||T_D(f_1, \cdots, f_d)|| &\leq C\, |\lambda|^{-\frac{1}{d}\delta} \ln^{d-\frac{1}{2}}(2 + |\lambda|) \prod_{i=1}^{d} ||f_i||_{\frac{d}{d-1}} \end{aligned} \tag{1.7}$$

Here $\delta$ is the Newton decay rate, defined to be $\alpha_1^{-1}$ if $\alpha = (\alpha_1, \cdots, \alpha_d)$ is the boundary point we just described. The decay rate $\frac{1}{d}\delta$ in (1.7) is an immediate consequence of Theorems A and B, since $\frac{1}{|\alpha|} = \frac{1}{d\alpha_1}$. Up to the logarithmic factors, Theorem B is then the natural multilinear generalization of the relation between decay rates for oscillatory integrals and Newton distances found in [8] for bilinear forms.

The oscillatory integral operators in [6–9] were defined with smooth cutoff functions, and satisfied sharp estimates without logarithmic terms. Similarly, some of the logarithmic factors in Theorem B can be eliminated if we use smooth cutoff functions. We formulate here a basic result of that type. In particular, for polynomial phases, we obtain at the same time a new proof of the estimates for oscillatory integral operators in [8], as well as a proof of the stability of such estimates.

**Theorem C**. (Oscillatory Integral Estimates with Smooth Cutoffs) *Let $\alpha^{(1)}, \ldots, \alpha^{(K)} \in \mathbf{N}^2 \setminus \{0\}$ and let $S(x, y) \in \mathbf{R}[x, y]$ be any polynomial of degree $n_S$ in two variables. Let $\psi \in C_0^2(U)$, and assume that*

$$|S^{(\alpha^{(k)})}(x, y)| > 1, \quad 1 \leq k \leq K \tag{1.8}$$

*on the support of $\psi$. Then the oscillatory integral operator $T_\psi$ defined by*

$$T_\psi f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \psi(x, y) f(y) dy, \tag{1.9}$$

*satisfies the estimate*

$$||T_\psi||_{L^2 \to L^2} \leq C|\lambda|^{-\frac{1}{2}\delta} \tag{1.10}$$

*Here $\delta$ is the Newton decay rate, defined as in (1.7) for the reduced Newton polyhedron $N^*(\alpha^{(1)}, \cdots, \alpha^{(K)})$ associated to the vertices $\alpha^{(k)}$, $1 \leq k \leq K$. The* *constant C depends only on the vertices* $\alpha^{(k)}$*, the degree* $n_S$ *of* $S(x, y)$*, and the cutoff function* $\psi$*.*

Our approach makes essential use of a decomposition procedure described in detail in Lemma 2.3 and the paragraphs which precede it. This procedure allows us to reduce systematically the analysis on algebraic domains to special types of domains (“curved trapezoids"), whose geometry is particularly simple. Another noteworthy component is Lemma 4.1, which is a more flexible version of the “curved-box lemmas" of [6–8]. By exploiting the Hardy-Littlewood maximal function, this version allows trapezoids of *varying* widths instead of curved boxes of *constant* width as in [6–8], and is crucial to the removal of logarithms in Theorem C.

It has been observed by A. Seeger that if upper bounds on $S(x)$ and its derivatives are imposed, then stability is already implicit in [13] for oscillatory integral operators in dimension two and loss of $\epsilon$ growth (c.f. Theorem B with $d = 2$). We note that for the class of functions $F(x)$ and $S(x)$ considered here, the estimates in Theorems A and B are more general: they do not require any upper bound on $S(x)$ or $F(x)$. This is also the case for the estimates in [2]. The lack of conditions on upper bounds turns out to be quite important in the inductive proof of Theorems A and B themselves, since $S(x)$ and $F(x)$ get multiplied there by an arbitrarily large number.

Finally, we note that the Newton polyhedron has long been known to encode well the singularities of a function [1]. In particular, for *generic* functions $S(x)$, the Newton distance corresponds to the decay rate of the *scalar* oscillatory integrals with phase $S(x)$ [19]. However, the genericity restriction makes this correspondence imperfect, and one can ask whether there are *analytic* notions which correspond without exceptions to the *geometric* notion of the Newton distance. Theorems A and B can be viewed as an affirmative answer to this question: the desired notions are the multilinear operators $W_D$ and $T_D$. The underlying reason is particularly compelling: the discrepancy between Newton distances and decay rates of scalar oscillatory integrals can be traced back to the fact that the former depends on the coordinate system, while the latter does not. On the other hand, multilinear operators fix the directions of the axes, and this turns out to eliminate exactly the excess freedom in the choice of coordinate systems which was responsible for the original discrepancy.

## 2. The multilinear level set operators

This section is devoted to the proof of Theorem A. We divide the proof into several steps. First, we treat the case of a single vertex in $\mathbf{N}^2$. Then the case of a single vertex in $\mathbf{N}^d$ is treated by induction on the dimension $d$. Finally, the general case of several vertices in $\mathbf{N}^d$ is treated by interpolation, establishing the desired estimate for an arbitrary vertex on the Newton polyhedron in $\mathbf{R}^d_{\geq 0}$. The case of a single vertex $(\alpha, \beta)$ in $\mathbf{N}^2$ requires the following preliminary lemma. This lemma is a consequence of Lemma 3.8 in [2], but we shall provide an independent proof for the convenience of the reader.

**Lemma 2.1** *Let $D \subset U$ be an open set. Let $F(x, y)$ be a smooth function on $D$ and assume that $|\partial_x^\alpha \partial_y^\beta F(x, y)| \geq 1$ and $|F(x, y)| \leq M$ for all $(x, y) \in D$. Then for every coordinate rectangle $R \subseteq D$ we have*

$$(\Delta x)^\alpha (\Delta y)^\beta \leq C \cdot M$$

*where $\Delta x$ and $\Delta y$ are the lengths of the sides of $R$, and $C$ depends only on $\alpha$ and $\beta$.*

*Proof.* If $f(t)$ is a function of one real variable, define $(\Delta f)(t) = f(t + \Delta x) - f(t)$. Then one easily sees that if $f$ is $k$ times differentiable,

$$\begin{aligned}
&\int_{x_0}^{x_0+\Delta x} \int_{x_1}^{x_1+\Delta x} \cdots \int_{x_{k-1}}^{x_{k-1}+\Delta x} f^{(k)}(t)\, dt dx_{k-1} dx_{k-2} \cdots dx_1 \\
&\quad = (\Delta^k f)(x_0) \\
&\quad = \sum_{m=0}^{k} (-1)^m \binom{k}{m} f(x_0 + m\Delta x)
\end{aligned} \tag{2.1}$$

In particular, if $f^{(k)}(t) \geq 1$ on the interval $[x_0, y_0]$ and if $|f| \leq M$ at the $k+1$ endpoints of the evenly spaced partition of that interval into $k$ subintervals, then

$$|y_0 - x_0| \leq 2k\, M^{1/k}$$

Let $(x_0, y_0)$ be the lower left corner of $R$. To prove the lemma, we apply (2.1) twice: First to the function $f(t) = \partial_y^\beta F(t, y)$, with $\Delta x$ replaced by $\Delta x/\alpha$, and then to the function $f(t) = F(x_0 + m\Delta x, t)$, with $\Delta y$ replaced by $\Delta y/\beta$. We obtain

$$(\Delta x)^\alpha (\Delta y)^\beta \leq \sum_{m=0}^{\alpha} \sum_{n=0}^{\beta} \binom{\alpha}{m} \binom{\beta}{n} |F(x_0 + m\Delta x, y_0 + n\Delta y)| \leq 2^{\alpha+\beta} \cdot M$$

**Step 1.** We consider the case where $D = U$ and $F(x, y) = x^\alpha y^\beta$. Let

$$D_M = \{(x, y) \in U : x > 0, y > 0, x^\alpha y^\beta < M\}$$

and let $W_D$ be the integral operator defined in (1.2)

$$W_D(f, g) = \int_{D_M} \chi_M(x, y) f(x) g(y)\, dx dy \tag{2.2}$$

We claim that $||W_D|| \leq CM^{1/(\alpha+\beta)}$ for some universal constant $C$. Let $N$ be the integer such that $2^{-N-1} < M \leq 2^{-N}$. Without loss of generality, we may assume that $M = 2^{-N}$. Let $D_i = \{(x, y) \in D_M : 2^{-N-i} < x^\alpha y^\beta \leq 2^{-N-i+1}\}$, $i = 1, 2, ...$, so that $W_D = \sum W_i$ where $W_i$ is defined by (2.2), but with the domain of integration $D_M$ replaced by $D_i$. It suffices, by the triangle inequality, to prove that

$$||W_i|| \leq C' 2^{(-N-i)/(\alpha+\beta)} \tag{2.3}$$

for some universal constant $C'$. For $k, l \geq 0$ let

$$R_i(k,l) = \{(x, y) \in D_i : 2^{-k-\frac{1}{2}} \leq x^\alpha \leq 2^{-k+\frac{1}{2}}, 2^{-l-\frac{1}{2}} \leq y^\beta \leq 2^{-l+\frac{1}{2}}, \}$$

Then the pairwise intersections of the rectangles $R_i(k, l)$ have measure 0, and the collection $\{R_i(k, l) : k + l = N + i - 1, \ \ k, l \geq 0\}$, covers $D_i$. Let $W_i(k, l)$ be defined by

$$W_i(k,l)(f, g) \ = \int_{R_i(k,l)} f(x)g(y)\, dxdy.$$

Now Hölder's inequality implies, for any dual pair $(p, q)$,

$$\begin{aligned} |W_i(k,l)(f,g)| &\leq \left(\int_{x^\alpha \sim 2^{-k}} |f(x)|dx\right)\left(\int_{y^\beta \sim 2^{-l}} |g(y)|dy\right) \\ &\leq 2^{(-\frac{k}{\alpha q} - \frac{l}{\beta p})}||f||_p||g||_q \end{aligned} \tag{2.4}$$

Taking $\frac{1}{q} = \frac{\alpha}{(\alpha+\beta)}$ and $\frac{1}{p} = \frac{\beta}{(\alpha+\beta)}$, we see that $||W_i(k, l)||$ is bounded by the expression appearing on the right side of (2.3). The desired estimate for $||W_i||$ follows from the following simple “almost-orthogonality" lemma, where we let $I_k = \{2^{-k-\frac{1}{2}} \leq x \leq 2^{-k+\frac{1}{2}}\}$, and $J_k = I_{N+i-1-k}$.

**Lemma 2.2**. *Let $T(f, g) = \int_{\Omega_1 \times \Omega_2} K(x, y) f_1(x) f_2(y) dxdy$ be a bilinear operator. Assume that*

$$\{(x, y); K(x, y) \neq 0\} \subset \cup_{k=1}^{\infty}(I_k \times J_k),$$

*where $I_k$ and $J_k$ are measurable subsets of $\Omega_1$ and $\Omega_2$ respectively, and $I_k \cap I_l$ and $J_k \cap J_l$ have zero measure for $k \neq l$. Let $\chi_{I_k}$, $\chi_{J_k}$ be the characteristic functions of the sets $I_k$ and $J_k$, and let $T_k$ be the bilinear operator with kernel $\chi_{I_k}(x)\chi_{J_k}(y)K(x, y)$. If $(p, q)$ is a pair of dual exponents $\frac{1}{p} + \frac{1}{q} = 1$, let $||T_k||$ and $||T||$ be the norms of $T_k$ and of $T$ as bilinear operators on $L^p(\Omega_1) \times L^q(\Omega_2)$. Then*

$$||T|| \leq \sup_k ||T_k||. \tag{2.5}$$

*Proof of Lemma 2.2*. Let $f, g$ be functions with $||f||_p = ||g||_q = 1$. Then we may write

$$|T(f, g)| = \left|\sum_k T_k(\chi_{I_k} f, \chi_{J_k} g)\right| \leq \sup_k ||T_k|| \sum_k ||\chi_{I_k} f||_p ||\chi_{J_k} g||_q$$

By Young's inequality, the sum in the above right hand side is bounded by

$$\sum_k \left( \frac{1}{p} \int_{I_k} |f|^p + \frac{1}{q} \int_{J_k} |g|^q \right) \leq \frac{1}{p} ||f||_p^p + \frac{1}{q} ||g||_q^q = 1,$$

establishing Lemma 2.2.

To describe the next several steps, we need the notion of a curved trapezoid.

**Definition 2.1** *Let $D \subseteq U$. Then $D$ is called a curved trapezoid if there exist real numbers $a, b$ with $a < b$ and continuous monotone functions $f, g : [a, b] \to \mathbf{R}$ with $f(x) > g(x)$ for all $x \in (a, b)$ such that*

$$D = \{(x, y) \in U : a < x < b, \text{ and } g(x) < y < f(x)\} \tag{2.6}$$

**Step 2.** We establish now the case where $|\partial_x^\alpha \partial_y^\beta F| > 1$ and $D$ is a curved trapezoid as in (2.6).

Assume first that $f'(x)g'(x) < 0$ for all $x$. In fact, without loss of generality, assume that $f'(x) < 0$ and $g'(x) > 0$ (the other possiblity is treated in an analogous way). Let $c = (f(b) + g(b))/2$. The line $y = c$ cuts $D$ into two pieces. After making the necessary translation to the origin, we may assume $c = 0$ and $a = 0$. Then the top piece is the region under the curve $y = f(x)$, above the $x$ axis, bounded on the left by the $y$ axis and on the right by $x = b$. Now Lemma 2.1 implies that $x^\alpha y^\beta \leq CM$ for all $x \in [0, b]$. Thus the estimate for the top piece follows from the case treated in Step 1. The estimate for the bottom piece is similar.

Now assume that $f'(x)g'(x) > 0$ for all $x$. Without loss of generality, we may assume $f'(x) < 0$ and $g'(x) < 0$ (the other possiblity is treated in an analogous way). Fix $x_0 \in (a, b)$. Then define, for $i \geq 0$, the following sequences of numbers:

$$y_0 = f(x_0),\ z_0 = x_0,\ w_0 = g(z_0),\ x_1 = f^{-1}(w_0),\ y_1 = f(x_1),\ z_1 = x_1, \ldots$$

$$z_{-1} = g^{-1}(y_0), w_{-1} = g(z_{-1}),\ x_{-1} = z_{-1},\ y_{-1} = f(x_{-1}),\ z_{-2} = g^{-1}(y_{-1}), \ldots$$

Then if we remove from $D$ all the vertical line segments whose endpoints are $(x_i, y_i)$ and $(z_i, w_i)$, and all the horizonal line segments whose endpoints are $(z_i, w_i)$ and $(x_{i+1}, y_{i+1})$, we are left with connected components which are curved right triangles (the two perpendicular sides are straight line segments, but the "hypotenuse" is a monotonic curve). The estimate for each triangle follows from the case treated in Step 1, and the estimate for all of $D$ follows from the almost-orthogonality Lemma 2.2.

**Step 3.** In this important step, we show that any algebraic domain $D$ can be decomposed into a finite, bounded number of curved trapezoids. We have seen how estimates for the level set operator $W_D$ can be obtained when the domain is a curved trapezoid. It may be helpful at this moment to stress the other important property of curved trapezoids, namely that they are both *horizontally convex* and *vertically convex*, in the following sense.

**Definition 2.2** *A set $V \in \mathbf{R}^2$ is horizontally convex if $(x, z) \in V$ and $(y, z) \in V$ implies that $(u, z) \in V$ for $u \in [x, y]$. The notion of vertical convexity is defined in the same way.*

The importance of these concepts will become apparent in Section III, when we discuss the oscillatory integral operators $T_D$. For the moment, we discuss how to decompose an algebraic domain $D$ into, say, horizontally convex components. The procedure is roughly as follows. Let $X$ denote the boundary of $D$. Then $X$ is a real algebraic curve. Let us call a real number $c \in (-1, 1)$ *critical* if the line $y = c$ intersects $X$ at a point of tangency, or at a point where $X$ is not smooth. Then, since the critical points are defined by algebraic equations whose degrees are bounded in terms of $n$, there are only a finite number of critical points and the number of such points is bounded in terms of $n$. Now consider the domain

$$D^* = \{(x, y) \in D : y \neq c \text{ for all critical values } c\}$$

Then the connected components of $D^*$ are all horizontally convex. Moreover, there are a finite number of connected components, bounded in terms of $n$.

Similarly, to decompose $D$ into curved trapezoids, we remove from $D$ all vertical lines $L$ which satisfy any one of the following properties:

1. $L$ intersects $X$ at a non-smooth point of $X$.
2. $L$ intersects $X$ at a point where the tangent line is vertical.
3. $L$ intersects $X$ at a point where the tangent line is horizontal.

Then the remaining domain has finitely many components, bounded in terms of $n$, each of which is a curved trapezoid. The precise construction is provided by the following lemma and its proof:

**Lemma 2.3** *Let $D$ by an algebraic domain of type $(r, n, 2, w)$. Then there exist curved trapezoids $\tau_1, ..., \tau_M \subseteq D$, and a set of measure zero $Z \subseteq D$ such that*

$$D = \left( \bigcup_{\mu=1}^{M} \tau_\mu \right) \cup Z \tag{2.7}$$

*where the union is disjoint and the number $M$ is bounded in terms of $r$, $n$, and $w$. In particular,*

$$\int_D f(x, y)\, dxdy = \sum_{\mu=1}^{M} \int_{\tau_\mu} f(x, y)\, dxdy$$

*for any measurable function $f$.*

*Proof of Lemma 2.3* We first observe that it suffices to prove the lemma in the case where $D$ is a simple algebraic domain. Indeed, if $D_1$ and $D_2$ are two simple algebraic domains

$$D_1 = \{x \in U;\ f_i(x) \geq A_i\}, \qquad D_2 = \{x \in U;\ g_j(x) \geq B_j\}$$

then $D_1 \cup D_2 = D_1 \cup \tilde{D}_2$, with $\tilde{D}_2$ given by

$$D_2 = \{x \in U;\ g_j(x) \geq B_j, \quad -f_i(x) \geq -A_i\}.$$

The intersection $D_1 \cup \tilde{D}_2$ has measure 0, and the type of $\tilde{D}_2$ is bounded in terms of the types of $D_1$ and $D_2$. Thus any algebraic domain can be rewritten as a union of simple algebraic domains whose pairwise intersections have measure 0.

Let $X \subseteq \mathbf{R}^2$ be an algebraic curve of degree at most $N$. This means that $X$ can be described as

$$X = \{(x, y) \in \mathbf{R}^2 :\ f(x, y) = 0\}$$

where $f(x, y) \in \mathbf{R}[x, y]$ has degree at most $N$.

We shall first establish the following:

**Claim.** There exist curved trapezoids $\tau_1, ..., \tau_{M'} \subseteq U \backslash X$ and a set $Z' \subseteq U \backslash X$ of measure zero such that

$$U \backslash X = \left( \bigcup_{\mu=1}^{M'} \tau_\mu \right) \cup Z' \tag{2.8}$$

Moreover, $M'$ is bounded interms of $N$.

We may assume that $f(x, y) = \prod_{i=1}^{m} f_i$ where the $f_i$ are irreducible and mutually relatively prime (that is, $f_i$ does not divide $f_j$ if $i \neq j$). Choose the notation so that $f_1, ..., f_a$ are functions of $x$ alone, and $f_{a+1}, ..., f_b$ are functions of $y$ alone, and $f_i(x, y)$, for $i > b$, is not independent of $x$ and is not independent of $y$. Assume further that if $b < i \leq c$, then $Z(f_i) = \{(x, y) : f_i(x, y) = 0\}$, the zero set of $f_i$, is finite (of course we may have $a = b = c = 0$).

For $i \neq j, i, j > a$, let $I_{i,j} = \{(x, y) \in \mathbf{R}^2 : f_i(x, y) = f_j(x, y) = 0\}$ be the set of intersection points. Since $f_i$ and $f_j$ are relatively prime, Bezout's theorem (see [4]) implies that $|I_{i,j}|$ is bounded above by $deg(f_i)deg(f_j)$. In particular, $|I_{i,j}|$ is bounded in terms of $N$.

For $i > c$, let $H_i = \{(x, y) \in \mathbf{R}^2 : f_i(x, y) = 0 \ \ and \ \ \frac{\partial f_i}{\partial x} = 0\}$ be the set of horizontal tangent points. Then again, Bezout's theorem implies that $|H_i|$ is bounded in terms of $N$. Similarly, if we let $V_i = \{(x, y) \in \mathbf{R}^2 : f_i(x, y) = 0 \ \ and \ \ \frac{\partial f_i}{\partial y} = 0\}$ be the set of vertical tangent points, then $|V_i|$ is bounded in terms of $N$.

Let $b < i \leq c$. Then, by assumption, $Z(f_i)$, the zero set of $f_i$, is finite. Moreover, the implicit function theorem implies that for every $(x, y) \in Z(f_i)$, we have $\frac{\partial f_i}{\partial x} = 0$ (and $\frac{\partial f_i}{\partial x} = 0$ as well). Thus, arguing as above, we must have $|Z(f_i)|$ is bounded in terms of $N$.

Finally, let $B = \{(x, y) \in \partial U : f_i(x, y) = 0 \text{ for some } i > b\}$. Note that $\partial U$ consist of four pieces, each of which is defined by polynomial of degree one (i.e., each is a line segment). Thus Bezout's theorem implies the intersection set has bounded cardinality, that is, $|B|$ is bounded in terms of $N$. (In this last case we don't really need Bezout's theorem: We are just making use of the fact that a non-zero polynomial in one variable has a finite number of roots, and that the number of roots is bounded by the degree of the polynomial).

Now we let

$$\Sigma = \left( \bigcup_{i,j>c} I_{i,j} \right) \cup \left( \bigcup_{i>c} H_i \right) \cup \left( \bigcup_{i>c} V_i \right) \cup \left( \bigcup_{b<i\leq c} Z(f_i) \right) \cup B$$

and we let $\Lambda_1 = \{(x, y) \in \mathbf{R}^2 : (x, y') \in \Sigma \text{ for some } y' \in \mathbf{R}\}$. In other words, $\Lambda_1$ consists of all vertical lines which intersect $\Sigma$ .

Let $\Lambda_2 = \{(x, y) : x = 1 \text{ or } x = -1\}$.

Let $\Lambda_3 = \{(x, y) : f_i(x, y) = 0 \text{ for some } i, 1 \leq i \leq a\}$.

Finally we let $Z' = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Then $Z'$ consists of $p$ vertical lines where $p$ is bounded in terms of $N$ (in fact $p \leq |\Sigma| + 2 + a$).

Let $\Omega = U \backslash \{X \cup Z'\}$. Then $\Omega$ is an open set. We must show that $\Omega$ consists of finitely many connected components, each of which is an open curved trapezoid, and that the number of such trapezoids is bounded in terms of $N$.

Note that $U \backslash Z'$ is a disjoint union of open rectangles, $R_1 \cup \cdots R_p$. Thus it suffices to prove for each $R = R_i$, that $R \backslash X$ is a bounded union of open trapezoids.

Choose $\alpha, \beta \in [0, 1]$ such that $R = \{(x, y) \in \mathbf{R}^2 : x \in (\alpha, \beta), y \in (-1, 1)\}$.

By definition, $R \cap X = \cup_{i=1}^m (Z(f_i) \cap R)$. Since $\Lambda_3 \cap R = \emptyset$, we have $R \cap X = \cup_{i=a}^m (Z(f_i) \cap R)$. If $a < i \leq b$, then $Z(f_i) \cap R$ is a horizontal line segment, that is, of the form: $Z(f_i) \cap R = \{(x, y) \in \mathbf{R}^2 : \alpha < x < \beta,\ y = d\}$.

If $b < i \leq c$, then $(Z(f_i) \cap R) = \emptyset$ by construction.

If $c < i \leq m$ then $(Z(f_i) \cap R)$ is a smooth curve with no horizontal or vertical tangents (since $H_i$ and $V_i$ do not intersect $R$). Thus each of its components is the graph of smooth monotone function, that is, each component is of the form:

$$\{(x, y) \in \mathbf{R}^2 : \alpha < x < \beta,\ y = \phi(x)\}$$

where $\phi : (\alpha, \beta) \to (-1, 1)$ is a smooth monotone function.

Thus we conclude that $R \cap X = \cup_{j=1}^{M'} \{(x, y) \in \mathbf{R}^2 : \alpha < x < \beta,\ y = \phi_j(x)\}$ where the $\phi_j(x)$ are smooth monotone functions on the interval $(\alpha, \beta)$. Moreover, the fact that $I_{i,j} \cap R = \emptyset$ for $i, j > a$ implies that $\phi_j(x) \neq \phi_k(x)$

for all $x \in (\alpha, \beta)$ and all $j \neq k$. Thus we may assume that the $\phi$ are ordered: $\phi_1 < \phi_2 \cdots < \phi_{M'}$. Define $\phi_0(x) = -1$ and $\phi_{M'+1}(x) = 1$. Now if we let $\tau_\mu = \{(x, y) : x \in (\alpha, \beta),\ \phi_\mu(x) < y < \phi_{\mu+1}(x)\}$ we obtain

$$R \backslash X = \bigcup_{\mu=0}^{M'} \tau_\mu$$

Moreover, if $\gamma \in (\alpha, \beta)$ then $M'$ is bounded by the number of intersections of $x = \gamma$ with $X$, that is $M$ is bounded by $N$. This completes the proof of the claim.

We now finish the proof of Lemma 2.3. Apply the claim to the algebraic curve $X$ which is defined by the equation $\prod_{k=1}^{r'}(f_k - A_k) = 0$. Then

$$U \backslash X = \left( \bigcup_{\mu=1}^{M'} \tau_\mu \right) \cup Z' \tag{2.9}$$

For each $\mu$, the functions $f_k - A_k$ all have constant sign on $\tau_\mu$. Choose the notation so that $f_k - A_k$ is positive on $\tau_\mu$ if and only if $\mu \leq M$. Then (2.7) readily follows from (2.9).

Choose $c_1 < \cdots < c_p$ such that $\Lambda \cap U = \{(x, y) : y \in (-1, 1)$ and $x = c_i$ for some $i\}$. Let $c_0 = -1$ and $c_{p+1} = 1$. Then each connected component of $\Omega$ is contained in $S_i = \{(x, y) : c_i < x < c_{i+1}$ and $y \in (-1, 1)\}$ for some $i$.

Now $\partial\Omega = (\Lambda \cap \bar{U}) \cup ((X \cap U)\backslash\Lambda) \cup (\partial U \backslash \Lambda)$ is a disjoint decomposition of the boundary of $\Omega$. The set $((X \cap U)\backslash\Lambda) \cap S_i$ consists of a finite number of components. These components are curves (and not points) since the $Z(f_i)$, $b < i \leq c$ have been removed. The curves do not intersect since the $I_{i,j}$, $i, j > c$ have been removed. The curves are the graphs of smooth monotone functions since the $H_i$ and $V_i$ have been removed for $i > c$. Moreover, by construction, the graphs of the smooth monotone functions are defined on the interval $(c_i, c_{i+1})$. Since these graphs do not intersect, they are ordered by heights, and the regions between two consecutive graphs is , by definition, a curved trapezoid. The number of such graphs is at most $N$ (since any vertical line can intersect $X$ at most $N$ times) and, as explained above, the number $p$ is also bounded in terms of $N$. Thus, each connected component is a trapezoid and the number of such trapezoids is bounded in terms of $N$. The proof of Lemma 2.3 is complete.

**Step 4**. Combining Steps 2 and 3, we have proved Theorem A in the case $d = 2$ with a single vertex. Assume now that $d > 2$: We shall proceed by induction on the dimension $d$, and write $\alpha = (\alpha', \alpha_d)$. We may assume that $\alpha_i > 0$ for all $1 \leq i \leq d$. Otherwise, if say $\alpha_d = 0$, then the relevant $L^{p_i}$ norms for $f_i$ are $p_d = 1$, and $p_i = \frac{|\alpha|}{|\alpha| - \alpha_i} = \frac{|\alpha'|}{|\alpha'| - \alpha_i}$ for $1 \leq i \leq d-1$. Thus the desired estimate follows immediately from the $(d-1)$-dimensional case.

Now let $D_j$ be the domain

$$D_j = \{x \in D : \ \ 2^{-j+1} \geq |F^{(\alpha',0)}(x)| \geq 2^{-j}\ \ \}$$

and let $D_0$ be the domain

$$D_0 = \{x \in D : \ |F^{(\alpha',0)}(x)| \geq 1 \ \}.$$

Then we may write

$$|W_{D_j}(f_1, ..., f_d)| \leq \int \left| \int \cdots \int_{D_j} \chi_{F \leq M}(x_1, ..., x_d) f_1(x_1) \cdots f_{d-1}(x)\, dx_1 \cdots dx_{d-1} \right| |f_d(x_d)| dx_d$$

where the domain of integration for the inner integral is $D_j(x_d)$, the cross section of $D_j$ obtained by setting the last variable equal to $x_d$. For each $x_d$, the inner integral can be viewed as a level set $(d-1)$-linear operator in dimension $d-1$, with $F(x)$ replaced by $2^j F(x)$, $M$ replaced by $2^j M$. This $(d-1)$-linear operator satisfies $|(2^j F(x))^{(\alpha',0)}| \geq 1$. Applying the induction hypothesis, we obtain the first estimate below

$$|W_{D_j}(f_1, ..., f_d)| \leq C \begin{cases} \left(2^j M\right)^{\frac{1}{|\alpha'|}} \ln^{d-3}\left(2 + \frac{1}{2^j M}\right) \cdot \prod_{i=1}^{d-1} ||f_i||_{p_i} \cdot ||f_d||_1, \\ \qquad \text{if} \quad 2^j M < 1; \\ \prod_{i=1}^{d-1} ||f_i||_{p_i} \cdot ||f_d||_1, \\ \qquad \text{if} \quad 2^j M \geq 1. \end{cases} \tag{2.10}$$

where $\frac{1}{q_i} = \frac{\alpha_i}{|\alpha'|}$. The second estimate is of course trivial.

On the other hand

$$|W_{D_j}(f_1, ..., f_d)| \leq \int \cdots \int \left\{ \int_{D_j(x_1, ..., x_{d-1})} |f_d(x_d)| dx_d \right\} \prod_{i=1}^{d-1} |f_i(x_i)| dx_1 \cdots dx_{d-1}$$

where $D_j(x_1, ..., x_{d-1})$ is the cross section of $D_j$ obtained by fixing $x_1, ..., x_{d-1}$. In view of the hypothesis $|F^{(\alpha)}(x)| > 1$, the cross-section $D_j(x_1, \cdots, x_{d-1})$ has measure $\leq C(2^{-j})^{\frac{1}{\alpha_d}}$, where $C$ is a constant depending only on $\alpha_d$ (This basic fact was established implicitly in the proof of Lemma 2.1. Another proof can be found in [3]). Thus

$$|W_{D_j}(f_1, ..., f_d)| \leq (2^{-j})^{\frac{1}{\alpha_d}} \prod_{i=1}^{d-1} ||f_i||_1 \cdot ||f_d||_\infty$$

We can now apply the following interpolation lemma for multilinear operators which is a straightforward extension of the usual bilinear Riesz-Thorin interpolation theorem (see e.g. [17], Chapter V):

**Lemma 2.4**. *Let $T(f_1, ..., f_d)$ be a multilinear operator such that:*

$$|T(f_1, ..., f_d)| \leq A \prod_{i=1}^{d} ||f_i||_{p_i}$$

*and*

$$|T(f_1, ..., f_d)| \leq B \prod_{i=1}^{d} ||f_i||_{\tilde{p}_i}$$

*Let $\theta \in [0, 1]$ and set $\frac{1}{r_i} = \theta \left(\frac{1}{p_i}\right) + (1-\theta)\left(\frac{1}{\tilde{p}_i}\right)$. Then*

$$|T(f_1, ..., f_d)| \leq A^{\theta} B^{1-\theta} \cdot \prod_{i=1}^{d} ||f_i||_{r_i}$$

To get estimates with respect to the norm $||f_1||_{\frac{|\alpha|}{|\alpha|-\alpha_1}}$, say, the interpolation parameter $\theta$ should satisfy

$$\left(\frac{|\alpha|}{|\alpha|-\alpha_1}\right)^{-1} = \theta\left(\frac{|\alpha'|}{|\alpha'|-\alpha_1}\right)^{-1} + (1-\theta)\cdot 1^{-1}$$

This gives $\theta = \frac{|\alpha'|}{|\alpha|}$, which implies $1-\theta = \frac{\alpha_d}{|\alpha|}$, and $p_i = \frac{|\alpha|}{|\alpha|-\alpha_i}$ for all $1 \leq i \leq d$. We may apply now Lemma 2.4 with this choice for $\theta$, $B = (2^{-j})^{\frac{1}{|\alpha_d}}$, and $A = (2^j M)^{\frac{1}{|\alpha'|}} \ln^{d-3}(\frac{1}{2^j M})$ or $A = 1$, depending on whether $2^j M < 1$ or $2^j M \geq 1$. The norm of $W_{D_j}$ is found to be bounded by

$$||W_{D_j}|| \leq C \begin{cases} M^{\frac{1}{|\alpha|}} \ln^{\theta(d-3)}\left(2 + \frac{1}{2^j M}\right), & \text{if } 2^j M < 1; \\ 2^{-\frac{j}{|\alpha|}}, & \text{if } 2^j M \geq 1. \end{cases}$$

Bounding $\theta(d-3)$ by $d-3$, and summing in $j$ gives the desired estimate. We note that this summation in $j$ is the only part giving rise to logarithmic terms in Theorem A.

**Step 5**. We can now prove Theorem A in its generality, with an arbitrary number of vertices. Let $\alpha \in \mathbf{R}^d_{\geq 0}$. We say that $\alpha$ is “established" if the inequality asserted in the statement of Theorem A is valid. To prove the theorem, it suffices to show:

A) If $\alpha$ is established then every element of $P(\alpha)$ is established.
B) The set of established vectors is convex.

We start with a proof of B): Assume that $\alpha$ and $\beta$ are established. Choose positive real numbers $\lambda_1, \lambda_2$ such that $\lambda_1 + \lambda_2 = 1$, and let $\gamma = \lambda_1\alpha + \lambda_2\beta$. We must show that $\gamma$ is established.

Let $\theta_1 = \lambda_1 \frac{|\alpha|}{|\gamma|}$, $\theta_2 = \lambda_2 \frac{|\beta|}{|\gamma|}$. Then it is clear that $\theta_1 + \theta_2 = 1$ and that $\theta_i \geq 0$. Moreover we have

$$\frac{\gamma}{|\gamma|} = \theta_1 \frac{\alpha}{|\alpha|} + \theta_2 \frac{\beta}{|\beta|} \quad \text{and} \quad \frac{1}{|\gamma|} = \theta_1 \frac{1}{|\alpha|} + \theta_2 \frac{1}{|\beta|}$$

Thus we may apply the interpolation Lemma 2.4 with $\theta = \theta_1$ to the estimates for the vertices $\alpha$ and $\beta$, establishing $\gamma$.

Now we prove assertion A): Assume that $\alpha = (\alpha_1, ..., \alpha_d)$ is established. Formally, the vertex $\beta = (\alpha', \alpha_d = \infty)$ is also established, in the sense that the trivial estimate

$$|W_D(f_1, \cdots, f_d)| \leq \prod_{i=1}^{d-1} ||f_i||_1 \cdot ||f_d||_\infty$$

always holds. Thus we can apply Lemma 2.4 to $(p_1, ..., p_d) = (\frac{|\alpha|-\alpha_1}{|\alpha|}, ..., \frac{|\alpha|-\alpha_d}{|\alpha|})$, $A = M^{1/|\alpha|}$, $(\tilde{p}_1, ..., \tilde{p}_d) = (1, \cdots, 1, \infty)$, $B = 1$. We conclude that $(\alpha', \frac{\alpha_d}{\theta} + (\frac{1}{\theta} - 1)|\alpha'|)$ is established for every $\theta \in (0, 1]$). But as $\theta$ ranges over $(0, 1]$, $\frac{\alpha_d}{\theta} + (\frac{1}{\theta} - 1)|\alpha'|$ ranges over $[\alpha_d, \infty)$. This proves that $(\alpha', \alpha_d + p)$ is established for every $p \geq 0$. Similarly, $(\alpha_1, \alpha_2, .., \alpha_i + p, \alpha_{i+1}, ..., \alpha_d)$ is established for every $i$ and every $p \geq 0$. Thus, by convexity, $P(\alpha)$ is established. This completes the proof of Theorem A.

## 3. The multilinear oscillatory integral operators

This section is devoted to the proof of Theorem B. Again, it is convenient to divide the proof into several steps.

**Step 1**. Here we establish the desired estimate in the simplest case, with $\alpha = (1, 1)$ in $\mathbf{R}^2_{\geq 0}$. In this case, we define the operator $T_D$ on $L^2(\mathbf{R})$ by

$$(T_D f)(y) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi_D(x, y) f(x) dx$$

where $\chi_D$ is the characteristic function of $D$. The desired estimate is

$$||T_D||_{L^2 \to L^2} \leq C|\lambda|^{-1/2} \ln^{1/2}(2 + |\lambda|). \tag{3.1}$$

The operator $T_D T_D^*$ is an integral operator with kernel $K(x, y)$ given by

$$K(x, y) = \int_{-\infty}^{\infty} e^{i\lambda(S(x,z) - S(y,z))} \chi_D(x, z) \chi_D(y, z)\, dz$$

For $x, y$ fixed, let

$$\Phi(z) = S'_z(x, z) - S'_z(y, z) = -\int_x^y S''_{uz}(u, z)\, du\ .$$

Let us assume first that the domain $D$ is horizontally convex (c.f. Definition 2.2). Then we would obtain the following estimate on $\Phi(z)$:

$$|\Phi(z)| \geq |x - y|$$

for all $x, y$ such that $\chi_D(x, z)\chi_D(y, z) \neq 0$.

Now, for $x, y$ fixed, the set of $z$ such that $\chi_D(x, z)\chi_D(y, z)\Phi'(z) \neq 0$ is a finite union of (possibly empty) intervals. The number of such intervals is bounded by a constant which depends only on the degree $n_S$ (since the endpoints of the intervals are roots of polynomials whose degrees are bounded in terms of $n_S$). Also, the number of intervals of monotonicity in $z$ of $\Phi'(z)$ is bounded uniformly in terms of $n_S$. Thus, by the standard van der Corput inequality

$$|K(x, y)| \ \leq \ \frac{C}{\lambda|x - y|}$$

On the other hand, $|K(x, y)| \leq 1$. It follows that

$$|K(x, y)| \ \leq \ \frac{C}{1 + \lambda|x - y|}$$

and

$$\int_{-\infty}^{\infty} |K(x, y)|\, dx \leq C \int_0^1 \frac{1}{1 + \lambda|x - y|} dx \leq C|\lambda|^{-1} \ln(2 + |\lambda|).$$

It follows that $||T_D T_D^*|| \leq C'|\lambda|^{-1} \ln(2 + |\lambda|)$, which establishes the desired inequality.

In view of Lemma 2.3, any algebraic domain $D$ can always be written as a disjoint union of boundedly many horizontally convex components, up to a set of measure 0. Thus the proof of the case $\alpha = (1, 1)$ in $\mathbf{R}^2_{\geq 0}$ is complete.

**Step 2**. Next, we consider a general central vertex of the form $\alpha = (l, l)$ in $\mathbf{R}^2_{\geq 0}$. We proceed by induction on $l$, the case $l = 1$ having already been established in Step 1. For this, we divide the domain $D$ into two algebraic domains

$$D' = \{(x, y) \in D; |S^{(l-1,l-1)}| \leq t\}, \qquad D'' = D \setminus D'$$

By the induction hypothesis, the operator $T_{D''}$ satisfies the estimate

$$||T_{D''}||_{L^2 \to L^2} \leq C|\lambda t|^{-\frac{1}{2(l-1)}} \ln^{1/2}(2 + |\lambda t|)$$

On the other hand, the norm of the operator $T_{D'}$ is clearly bounded by the norm of the operator $W_{D'}$ of (1.2), where we have set $F = S^{(l-1,l-1)}$ and $M = t$. Since we have then $|\partial_x \partial_y F| > 1$, Theorem A implies $||T_{D'}|| \leq ||W_{D'}|| \leq Ct^{1/2}$. Choosing $t = |\lambda|^{-1/l}$ establishes the desired estimate in the case $\alpha = (l, l)$ in $\mathbf{R}^2_{\geq 0}$.

**Step 3**. Now we consider the case for a general vertex in $\mathbf{R}^2_{\geq 0}$. Say $\alpha = (l+m, l)$ with $l > 0$. We divide the domain $D$ into domains $D_j$

$$D_j = \{(x,y) \in D; 2^{-j-\frac{1}{2}} \leq |S^{(l,l)}(x,y)| < 2^{-j+\frac{1}{2}}\}, \qquad j = 1, 2, \cdots$$
$$D_0 = \{(x,y) \in D; |S^{(l,l)}(x,y)| \geq 2^{\frac{1}{2}}\}.$$

Let $T_{D_j}$ be the operators defined by (1.2), but with the range of integration $D$ replaced by $D_j$. We may apply the estimates already established for the vertex $(l,l)$ and obtain

$$||T_{D_j}||_{L^2\to L^2} \leq \begin{cases} C(2^{-j}\lambda)^{-\frac{1}{2l}} \ln^{1/2}(2+2^{-j}|\lambda|), & \text{if } 2^{-j}|\lambda| > 1; \\ 1, & \text{if } 2^{-j}|\lambda| \leq 1. \end{cases}$$

On the other hand, for each $y$, the cross-section $\{x; (x,y) \in D_j\}$ is of size $\leq C(2^{-j})^{1/m}$. It follows that

$$||T_{D_j}||_{L^\infty \to L^\infty} \leq 2^{-j/m}$$

To obtain the norm $||T_{D_j}||_{L^{\frac{2l+m}{l}} \to L^{\frac{2l+m}{l}}}$, we interpolate using Lemma 2.4 with $\theta = \frac{2l}{2l+m}$. The resulting estimates are

$$||T_{D_j}||_{L^{\frac{2l+m}{l}} \to L^{\frac{2l+m}{l}}} \leq C \begin{cases} |\lambda|^{-\frac{1}{2l+m}} \ln^{\theta/2}(2+|\lambda|), & \text{if } 2^{-j}|\lambda| > 1; \\ 2^{-\frac{j}{2l+m}}, & \text{if } 2^{-j}|\lambda| \leq 1. \end{cases}$$

We bound again $\theta/2$ simply by $1/2$, and sum in $j$ to obtain the desired estimate.

**Step 4**. We can now establish Theorem B in the case of a general single vertex $\alpha = (\alpha_1, \cdots, \alpha_d)$ in $\mathbf{R}^d_{\geq 0}$. As in the proof of Theorem A, we proceed by induction on the dimension $d$. The arguments here are similar to the induction step in Theorem A, so we shall be brief. Since $\alpha$ is a vertex in the reduced Newton diagram of $S(x_1, \cdots, x_d)$, we may assume that $\alpha = (\alpha', \alpha_d)$ with $|\alpha'| > 0$ and $\alpha_d > 0$. Set

$$D_j = \{(x,y) \in D; 2^{-j-\frac{1}{2}} \leq |S^{(\alpha',0)}(x,y)| < 2^{-j+\frac{1}{2}}\}, \qquad j = 1, 2, \cdots$$
$$D_0 = \{(x,y) \in D; |S^{(\alpha',0)}(x,y)| \geq 2^{\frac{1}{2}}\}.$$

Then the multilinear oscillatory integral operator is decomposed correspondingly into $T_D = \sum_j T_{D_j}$, with the following two sets of estimates

$$|T_{D_j}(f_1, \cdots, f_d)| \leq C \begin{cases} (2^{-j}|\lambda|)^{-\frac{1}{|\alpha'|}} \ln^{d-\frac{3}{2}}(2+2^{-j}|\lambda|) \prod_{i=1}^{d-1} ||f||_{\frac{|\alpha'|}{|\alpha'|-\alpha_i}} ||f_d||_1, \\ \qquad \text{if} \quad 2^{-j}|\lambda| > 1; \\ \prod_{i=1}^{d-1} ||f||_{\frac{|\alpha'|}{|\alpha'|-\alpha_i}} ||f_d||_1, \\ \qquad \text{if} \quad 2^{-j}|\lambda| \leq 1. \end{cases}$$

$$|T_{D_j}(f_1, \cdots, f_d)| \leq C 2^{-\frac{j}{\alpha_d}} \prod_{i=1}^{d-1} ||f_i||_1 ||f||_\infty$$

As before, the first set of estimates is a consequence of the induction hypothesis, while the second is a consequence of the fact that the $x_d$ cross-section of $D_j$ is of size $\leq C2^{-\frac{j}{\alpha_d}}$ for all $(x_1, \cdots, x_{d-1})$. Interpolating with $\theta = \frac{|\alpha'|}{|\alpha|}$ as before gives the estimates

$$|T_{D_j}(f_1, \cdots, f_d)| \\ \leq C \begin{cases} |\lambda|^{-\frac{1}{|\alpha|}} \ln^{\theta(d-\frac{3}{2})}(2 + 2^{-j}|\lambda|) \prod_{i=1}^d ||f||_{\frac{|\alpha|}{|\alpha|-\alpha_i}}, & \text{if } 2^{-j}|\lambda| > 1; \\ 2^{-\frac{j}{|\alpha|}} \prod_{i=1}^d ||f||_{\frac{|\alpha|}{|\alpha|-\alpha_i}}, & \text{if } 2^{-j}|\lambda| \leq 1. \end{cases}$$

Summing in $j$ gives now the estimate for a general vertex $\alpha$ in $\mathbf{R}^d_{\geq 0}$.

**Step 5**. We can now establish the general statement about reduced Newton polyhedra with several vertices just as in the last step of the proof of Theorem A. The set of vertices for which the estimate can be established is first shown to be convex by interpolation. The unbounded faces of the reduced Newton polyhedra are then also established by interpolating a given vertex $\alpha = (\alpha', \alpha_d)$ with formal vertices of the form $(\alpha', \infty)$, for which the estimates for $T_D$ trivially hold. The proof of Theorem B is complete.

## 4. Smooth cutoffs and logarithm removal

In this section, we provide the proof of Theorem C. A key ingredient is the following generalization of the "curved box" lemma of [6–8] to curved trapezoids, exploiting the Hardy-Littlewood maximal function.

**Lemma 4.1** (Curved Trapezoid Lemma) *Let $\tau$ be a curved trapezoid, i.e, a region of the form*

$$\tau = \{(x, y) \in U \in \mathbf{R}^2 : c < x < d \text{ and } g(x) \leq y \leq f(x)\}, \tag{4.1}$$

*where $f$ and $g$ are smooth, monotone functions on the interval $[c, d]$. Denote by $\tau(x)$ the cross-section $\{y; (x, y) \in \tau\}$, and let $\delta(x)$ be the length of $\tau(x)$. Let $\psi \in C_0^2(U)$ be a cutoff function, and assume that*

(i) $\mu \leq |S''_{xy}| \leq A\,\mu$ *on* $\tau$*;*
(ii) *For each $c < x < d$, the function $\psi(x, \cdot)$ is supported in the interval $g(x) \leq y \leq f(x)$;*
(iii) $|\partial_y^n \psi(x, y)| \leq B\,\delta(x)^{-n}$, *for* $(x, y) \in \tau$, $0 \leq n \leq 2$.

*Let $T_{\tau;\psi}$ be the oscillatory integral operator defined by*

$$T_{\tau;\psi}(h)(x) = \int e^{i\lambda S(x,y)}\psi(x,y)\chi_\tau(x,y)h(y)\,dy \tag{4.2}$$

*where $\chi_\tau(x,y)$ is the characteristic function of $\tau$. Then $T_{\tau;\psi}$ is a bounded operator on $L^2(\mathbf{R})$ with norm bounded by*

$$||T_{\tau;\psi}|| \le C|\mu\lambda|^{-\frac{1}{2}}, \tag{4.3}$$

*where the constant $C$ depends only on the degree $n_S$ of $S(x,y)$, and the constants $A$ and $B$ in* (i) *and* (iii).

*Proof of Lemma 4.1.* As in the last part of the argument for Step 2 in Section 2, by removing a countable number of vertical lines passing by the points $(x_i, y_i)$ and $(z_i, w_i)$, we can decompose the trapezoid $\tau$ into a union of trapezoids $\tau_i$, up to a set of measure 0. These trapezoids have pairwise empty intersections. So do their projections on the $x$ and the $y$ axes. If we decompose accordingly the operator $T_{\tau;\psi} = \sum_i T_{\tau_i;\psi}$ into operators $T_{\tau_i;\psi}$, it follows from the almost-orthogonality Lemma 2.2 that it suffices to prove the estimate (4.3) for each $T_{\tau_i;\psi}$. Thus we drop the index $i$, and assume from the outset that the boundary of $\tau$ satisfies $f(x_1) \ge g(x_2)$ for all $x_1, x_2 \in (c,d)$. By a suitable translation, we may assume actually that

$$f(x_1) \ge 0 \ge g(x_2) \quad \text{for all} \quad x_1, x_2 \in (c,d) \tag{4.4}$$

As in the proof of Step 1 in Section III, we follow now the proof of Lemma 1.1 in [6]. The kernel $K(x,y)$ of the operator $T_{\tau;\psi}T^*_{\tau;\psi}$ is given by

$$K(x,y) = \int_{-\infty}^{\infty} e^{i\lambda(S(x,z)-S(y,z))}\psi(x,z)\bar{\psi}(y,z)\chi_\tau(x,z)\chi_\tau(y,z)\,dz \tag{4.5}$$

Since the amplitude $\psi(x,z)\bar{\psi}(y,z)\chi_\tau(x,z)\chi_\tau(y,z)$ is $C^2$ in $z$ and has compact support for all $(x,y)$, we may integrate by parts in $z$ and write

$$\begin{aligned} &K(x,y) \\ &= \int_{-\infty}^{\infty} e^{i\lambda(S(x,z)-S(y,z))}(L^t)^2(\psi(x,z)\bar{\psi}(y,z)\chi_\tau(x,z)\chi_\tau(y,z))\,dz \end{aligned} \tag{4.6}$$

where $L^tF(z) = -(i\lambda)^{-1}\partial_z((\Phi(z))^{-1}F(z))$ and $\Phi(z) = S'_z(x,z) - S'_z(y,z)$. For $x, y \in [c,d]$ fixed we define $a(x,y) \in \mathbf{R}$, $b(x,y) \in \mathbf{R}$ and $\delta(x,y) \in \mathbf{R}$ by the formulas:

$$\begin{aligned} [a(x,y), b(x,y)] &= \{z \in \mathbf{R} : (x,z) \in \tau \text{ and } (y,z) \in \tau\}, \\ \delta(x,y) &= b(x,y) - a(x,y) \end{aligned} \tag{4.7}$$

Then we claim that $K(x,y)$ satisfies the estimates

$$|K(x,y)| \le C\delta(x,y)(1 + (\lambda\mu)^2\delta(x,y)^2|x-y|^2)^{-1}. \tag{4.8}$$

To see this, we note that the integrand in (4.5) is bounded, and the interval of integration is $[a(x,y), b(x,y)]$. Thus $|K(x,y)| \le C\delta(x,y)$. On the other hand, the hypothesis (iii) implies that

$$
\begin{aligned}
|\partial_z^n(\psi(x,z)\bar{\psi}(y,z)\chi_\tau(x,z)\chi_\tau(y,z))| &\le B\,[\min(\delta(x),\delta(y))]^{-n} \\
&\le B\,\delta^{-n}(x,y), \quad 0 \le n \le 2 \quad (4.9)
\end{aligned}
$$

We may write $S_z'(x,z) - S_z'(y,z) = (x-y)\int_0^1 S_{xz}''(x+u(y-x),z)du$. Since $S_{xz}''(x+u(y-x),z)$ does not change sign and is of constant size $\mu$, it follows that

$$|S_z'(x,z) - S_z'(y,z)| \sim \mu|x-y|$$

Furthermore,

$$\left|\partial_z \frac{1}{S_z'(x,z) - S_z'(y,z)}\right| = \frac{1}{|x-y|}\frac{|\int_0^1 S_{xzz}'''(x+u(y-x),z)du|}{|\int_0^1 S_{xz}''(x+u(y-x),z)du|^2} \tag{4.10}$$

For each $0 \le u \le 1$, the point $(x+u(y-x),z)$ is on a vertical interval of length $\delta(x+u(y-x))$, on which the polynomial $S_{xz}''(x+u(y-x),z)$ is of size $A\mu > 0$. It follows that

$$|S_{xzz}'''(x+u(y-x),z)| \le C\,\delta(x+u(y-x))^{-1}\mu \le C\,\delta(x,y)^{-1}\mu \tag{4.11}$$

where $C$ is a contant depending only on the degree $n_S$ of the polynomial $S(x,y)$ (see e.g. Lemma 1.2 in [6]). We obtain

$$\left|\partial_z \frac{1}{S_z'(x,z) - S_z'(y,z)}\right| \le C\,\delta^{-1}(x,y)(\mu|x-y|)^{-1} \tag{4.12}$$

The second derivative of $S_z'(x,z) - S_z'(y,z)$ can be estimated in the same way. Combining the estimates (4.9) and (4.12) in the expression (4.6) for $K(x,y)$, we find also that $|K(x,y)| \le C\delta(x,y)^{-1}(\mu|\lambda||x-y|)^{-2}$. This establishes (4.8).

Next, we require the following lemma:

**Lemma 4.2** *Suppose $K(x,y)$ is a kernel on $\mathbf{R}\times\mathbf{R}$ which satisfies*

$$|K(x,y)| \le C\delta(1+(\lambda\mu)^2\delta^2|x-y|^2)^{-1}, \tag{4.13}$$

*where $\delta = \delta(x)$ is a function of $x$ alone. Then there is a constant $C'$ such that*

$$\left|\int K(x,y)h(x)k(y)\,dxdy\right| \le C'\cdot(\lambda\mu)^{-1}||h||_2||k||_2$$

*for all $h, k \in L^2$.*

*Remarks.*

1. By symmetry, the same conclusion holds if $\delta$ is a function of $y$ alone.

2. The same conclusion holds if the right side of (4.13) is replaced by a finite sum of terms of the form $C\delta(1+(\lambda\mu)^2\delta^2|x-y|^2)^{-1}$ where in each term, $\delta$ is either a function of $x$ alone or a function of $y$ alone.

*Proof of Lemma 4.2.* Let $U(k)(x) = \int K(x,y)k(y)dy$. Then

$$|Uk(x)| \leq C \cdot (\lambda\mu)^{-1}\, Mk(x),$$

where $Mk(x)$ is the Hardy-Littlewood maximal function (see e.g. [16] and references therein). Since $k \to M(k)$ is bounded in the $L^2$ norm, Lemma 4.2 is proved.

Now we apply Lemma 4.2 to the kernel $K(x,y)$ defined by (4.5) and satisfying the estimate (4.8). The distance $\delta(x,y)$ can be written more explicitly, using the property (4.4) for the trapezoid $\tau$. Depending on whether $f(x)$ and $g(x)$ are increasing or decreasing, we have the following possibilities:

If both $f$ and $g$ are decreasing, then

$$[a(x,y),b(x,y)] \\ = \begin{cases}[g(y),f(x)], & \text{if } x \geq y;\\ [g(x),f(y)], & \text{if } y \geq x;\end{cases} \quad \delta(x,y) = \begin{cases}|f(x)|+|g(y)| & \text{if } x \geq y;\\ |f(y)|+|g(x)| & \text{if } y \geq x;\end{cases} \tag{4.14}$$

If $f$ is decreasing and $g$ is increasing, then

$$[a(x,y),b(x,y)] \\ = \begin{cases}[g(x),f(x)], & \text{if } x \geq y;\\ [g(y),f(y)], & \text{if } y \geq x;\end{cases} \quad \delta(x,y) = \begin{cases}|f(x)|+|g(x)| & \text{if } x \geq y;\\ |f(y)|+|g(y)| & \text{if } y \geq x;\end{cases} \tag{4.15}$$

The other cases are similar. We first consider the case where $f$ and $g$ are both decreasing. Then (4.14) implies that for $x \geq y$ we have

$$|K(x,y)| \leq C\frac{|f(x)|+|g(y)|}{1+(\lambda\mu)^2(|f(x)|+|g(y)|)^2|x-y|^2} \\ \leq \frac{|f(x)|}{1+(\lambda\mu)^2|f(x)|^2|x-y|^2} + \frac{|g(y)|}{1+(\lambda\mu)^2|g(y)|^2|x-y|^2}$$

For $y \geq x$, we have the same estimate, but with $x$ and $y$ reversed. Thus Lemma 4.2 implies that the desired estimate (4.3) holds in this case.

Next consider the case where $f$ is decreasing and $g$ is increasing. In this case (4.15) implies that $\delta(x,y)$ is a function of $x$ alone if $x \geq y$ and a function of $y$ alone if $y \geq x$. Thus Lemma 4.2 implies that (4.3) holds in this case as well. The remaining cases are analogous. Thus we conclude that (4.3) holds in all cases and Lemma 4.1 is proved.

*Proof of Theorem C.* We may assume that $\delta > 1$, since the case $\delta = 1$ is the non-degenerate case, which is simpler and well-known. Choose a positive smooth function $\rho : \mathbf{R} \to \mathbf{R}$ with the following properties:

1. $\rho(x) \neq 0$ if and only if $x \in (1,4)$.
2. $\sum_{j=-\infty}^{\infty} \rho(2^j x) = 1$ for all $x > 0$.

Then $T_{\tau;\psi} = \sum_{j=-\infty}^{\infty} T^{(j)}$ where

$$T^{(j)}(h)(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \psi_j(x,y) h(y)\, dy$$

and $\psi_j(x,y) = \psi(x,y)\rho(2^j |S''_{xy}(x,y)|)$.

It suffices to show that the operators $T^{(j)}$ satisfy the following two estimates (all norms are norms for operators from $L^2$ to $L^2$)

$$||T^{(j)}|| \le C\ \frac{1}{1 + 2^{\frac{j\delta}{2(1-\delta)}}}$$
$$||T^{(j)}|| \le C\ (2^{-j}|\lambda|)^{-\frac{1}{2}} \tag{4.16}$$

Indeed, (4.16) implies

$$||T_{\tau,\psi}|| \le \sum_{j=-\infty}^{\infty} \min\left( \frac{1}{1 + 2^{\frac{j\delta}{2(1-\delta)}}}, (2^{-j}|\lambda|)^{-\frac{1}{2}} \right) \le C\ |\lambda|^{-\frac{1}{2}\delta}$$

which is the desired estimate.

Now the first estimate in (4.16) follows from Theorem A, applied to the level set function $F(x,y) = S''_{xy}(x,y)$. Since the Newton diagram of $F(x,y)$ is just the reduced Newton diagram of $S(x,y)$ translated by the vector $(-1,-1)$, the Newton distance $\tilde{\delta}$ for $F(x,y)$ is related to the Newton decay rate for $S(x,y)$ by $\delta^{-1} = \tilde{\delta}^{-1} + 1$. Thus Theorem A implies that $||T^{(j)}|| \le C\, 2^{-\frac{j\delta}{2(1-\delta)}}$. Combining it with the trivial estimate $||T^{(j)}|| \le 1$ gives the first estimate in (4.16).

To establish the second estimate in (4.16), we observe that the support of the cutoff function $\psi_j(x,y)$ is a compact subset of the open algebraic domain

$$D = \{(x,y) \in U;\ \mu/2 < |S''_{xy}(x,y)| < 8\mu\} \tag{4.17}$$

with $\mu = 2^{-j}$. In particular, $\psi_j$ and all its partial derivatives vanish on $\partial D$.

By the decomposition Lemma 2.3, after removing a finite number of vertical lines, we may express $T^{(j)}$ as a boundedly finite sum of operators of the same form, but with cutoff function

$$\psi_j(\tau; x, y) = \psi_j(x,y)\chi_\tau(x,y)$$

where $\chi_\tau$ is the characteristic function of a curved trapezoid of the form (4.1). Using again the fact that $\max_{y\in I}|\partial_y^n S''_{xy}(x,y)| \le C\ |I|^{-n}\max_{y\in I}|S''_{xy}(x,y)|$ for any interval $I$ and any polynomial $S(x,y)$ of bounded degree (c.f. Lemma 1.2 in [6]), we readily verify that $\psi_j(\tau; x, y)$ satisfies the conditions (ii) and (iii) of Lemma 4.1 (it is not necessary to chop the domain horizontally). Thus Lemma 4.1 applies, giving the second estimate in (4.16). The proof of Theorem C is complete.

## 5. Concluding remarks

1. In their work [2], Carbery, Christ, and Wright applied $L^p$ estimates for level set operators to derive sharp bounds for the growth of the distribution functions when uniform lower bounds on a *single* vertex $\alpha$ are imposed. Naively, one may have hoped that these bounds could perhaps be improved if one imposed conditions on *several* vertices $\alpha = \alpha^{(1)}, \cdots, \alpha^{(K)}$. But this is not the case, and the bounds in [2] are always sharp in the context of the more global form of stability (c.f. Introduction). More precisely, the uniform growth rate $\gamma$ of the distribution function $|\{x \in U; |F(x)| \leq M\}| \leq CM^\gamma$ (up to logarithmic terms) over *all* functions $F(x)$ satisfying

$$\left|F^{(\alpha^{(1)})}\right| > 1, \cdots, \left|F^{(\alpha^{(K)})}\right| > 1 \tag{5.1}$$

is just

$$\gamma = \frac{1}{\min_{1\leq k\leq K}|\alpha^{(k)}|}. \tag{5.2}$$

This rate is in fact attained for $F(x) = \sum_{k=1}^{K}(x_1 + \cdots + x_d)^{|\alpha^{(k)}|}$. This behavior may be contrasted with that of individual phase functions, whose distribution functions do grow according to Newton distances in suitable coordinate systems (see e.g. [10]). The point here is of course the difference between the two notions of stability (or uniformity) explained in the Introduction.

2. Up to the logarithmic terms, all our estimates are sharp.

3. We have not striven to extract the smallest powers possible for the logarithmic terms which appear in Theorems A and B. At the least, they can be improved by powers depending on the vertex $\alpha$. For example, for $\alpha = (1, \cdots, 1)$ in $\mathbf{R}^d_{\geq 0}$, our arguments produce the estimate $|\lambda|^{-\frac{1}{d}} \ln^{\frac{1}{2}(1+d)-\frac{2}{d}}(2+|\lambda|)$. It would be interesting to determine for which classes of phase $S(x)$ or level set function $F(x)$ the logarithmic terms are necessary, if at all.

4. We stated the theorems in the category of polynomials, but the arguments go through verbatim in the category of smooth functions with some finiteness conditions. In dimension $d = 2$, it is clearly sufficient to assume that the boundary of the domains $D$ as well as the level sets of $F(x)$ and $S(x)$ and their derivatives are smooth with a boundedly finite number of singular points and turning points. The correct classes of domains and phases $S(x)$, $F(x)$ can then be defined recursively on the dimension, by requiring that any cross-section satisfies the conditions in one-dimension lower.

5. It is highly likely that the methods of the present paper can be adapted to establish stability results for damped estimates, such as those established in [9] for dimension $d = 2$. We shall return to this issue elsewhere [11].

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