Invent. math. 173, 651–665 (2008)
DOI: 10.1007/s00222-008-0134-x

Inventiones mathematicae

The Kähler–Ricci flow with positive bisectional curvature*

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Oblatum 5-VII-2007 & 1-IV-2008
Published online: 25 April 2008 –

Abstract. We show that the Kähler–Ricci flow on a manifold with positive first Chern class converges to a Kähler–Einstein metric assuming positive bisectional curvature and certain stability conditions.

1 Introduction

Let X be a compact Kähler manifold of complex dimension n with $c_1(X) > 0$. The Frankel conjecture, proved by Mori [Mor] and Siu–Yau [SY], states that if X admits a Kähler metric of positive bisectional curvature then it is biholomorphic to $\mathbf{P}^n$. There has been much interest in obtaining a proof of this using the Kähler–Ricci flow:

$$\frac{\partial}{\partial t} g_{\bar{k}j} = g_{\bar{k}j} - R_{\bar{k}j}. \tag{1.1}$$

By a result of Goldberg–Kobayashi [GK], this amounts to solving the following well-known 'folklore' problem: without using the existence of a Kähler–Einstein metric, show that if a Kähler metric has positive bisectional curvature then the Kähler–Ricci flow deforms it to a Kähler–Einstein metric.

* Research supported in part by National Science Foundation grants DMS-02-45371, DMS-06-04805, DMS-05-14003, and DMS-05-04285.

We mention now some work related to this problem. The case $n = 1$ was settled by Hamilton [H1], Chow [Cho] (see also Chen–Lu–Tian [CLT]). Bando [B] and Mok [Mok] showed that, in every dimension, the positivity of the bisectional curvature is preserved along the Kähler–Ricci flow. Chen–Tian [CT] used the Moser–Trudinger inequalities [T1,TZ1] (see also [PSSW1]) to show that if there exists a Kähler–Einstein metric then, starting at a metric with positive bisectional curvature, the flow converges to it. Perelman later showed, without any curvature conditions, that the flow converges to a Kähler–Einstein metric when one exists, and this was extended to Kähler–Ricci solitons by Tian-Zhu [P2,TZ2]. Using an injectivity radius estimate of Perelman [P1], Cao–Chen–Zhu [CCZ] showed that if the bisectional curvature is nonnegative then the Riemann curvature tensor is bounded along the flow.

In [PS3], it was shown that the folklore problem can be reduced to establishing various stability conditions. In this paper we succeed in making further progress along these lines. We consider the following three conditions:

- (A) The Mabuchi K-energy is bounded below on the space of Kähler metrics in the class $\pi c_1(X)$;
- (A') The Futaki invariant of X is zero;
- (B) Let J be the complex structure of X, viewed as a tensor. Then the C^∞ closure of the orbit of J under the diffeomorphism group of X does not contain any complex structure J_∞ with the property that the space of holomorphic vector fields with respect to J_∞ has dimension strictly higher than the dimension of the space of holomorphic vector fields with respect to J.

Conditions (A) and (A') and their relations to stability have been studied intensely in the last two decades, and for the definitions we refer the reader to the literature (see [PS1], for example). Condition (B) was introduced in [PS3]. It was shown there that if the curvatures along the Kähler–Ricci flow are uniformly bounded, and if (A) and (B) hold then the Kähler–Ricci flow converges exponentially fast to a Kähler–Einstein metric. Note that the Riemann curvature tensor is bounded along the flow if the bisectional curvature is nonnegative (this is explained in the proof of Lemma 4 below) or, in the case of complex dimension 2, if we have the weaker condition of nonnegative Ricci curvature with traceless curvature operator 2-nonnegative [PS2].

Our first result is as follows:

Theorem 1 *Suppose there exists a Kähler metric g_0 on X with nonnegative bisectional curvature which is positive at one point. Assume Condition (A) holds. Then the Kähler–Ricci flow starting at g_0 converges exponentially fast in C^∞ to a Kähler–Einstein metric.*

Now, at least *a priori*, the algebraic Condition (A') is much weaker than (A). Here, we strengthen the result of [PS3] by replacing (A) by Condition (A').

Theorem 2 *Suppose that the Riemann curvature tensor is uniformly bounded along the Kähler–Ricci flow and that Conditions (A') and (B) hold. Then the Kähler–Ricci flow converges exponentially fast in C^∞ to a Kähler–Einstein metric.*

If $n \leq 2$ we have:

Theorem 3 *Assume X has complex dimension 1 or 2, g_0 has nonnegative bisectional curvature and Condition (A') holds. Then the Kähler–Ricci flow starting at g_0 converges exponentially fast in C^∞ to a Kähler–Einstein metric.*

This result for $n = 1$ has already been established by different methods as mentioned above. Theorem 3 now shows that the folklore problem in complex dimension 2 can be reduced to a condition on the finite dimensional space of holomorphic vector fields.

We remark that there are already proofs of Theorems 1 and 3 which first show the existence of a Kähler–Einstein metric and then apply the results of [CT], [P2]. Indeed, Chen [Che] proved Theorem 1 by showing that the bisectional curvature along the flow approaches that of the Fubini-Study metric, concluding that the manifold is $\mathbf{P}^n$, and then applying [CT]. A proof of Theorem 3 can be obtained by combining [P2] with the result that, in complex dimension 2, the vanishing of the Futaki invariant implies the existence of a Kähler–Einstein metric [T1]. We note that our proofs use primarily flow methods and in particular avoid showing first the existence of a Kähler–Einstein metric.

A key step in the proofs of Theorems 1, 2 and 3 is to obtain a uniform lower bound for the first positive eigenvalue λ of the $\bar\partial^\dagger\bar\partial$ operator on $T^{1,0}$ vector fields. The idea of considering this eigenvalue along the Kähler–Ricci flow was introduced in [PS3] and examined further in [PSSW2]. In Sect. 2 we show that certain curvature conditions imply the desired bound for λ. In Sects. 3, 4 and 5, we give the proofs of Theorems 1, 2 and 3 respectively.

Remark In Theorem 2, the conclusion still holds if we replace (B) by the following condition: the eigenvalue λ is uniformly bounded from below along the flow. Assuming bounded curvature along the flow, this condition is *a priori* weaker than (B).

2 Lower bounds for the $\bar\partial$ operator

For a solution $g(t)$ of the Kähler–Ricci flow (1.1), we define the Ricci potential u by $\frac{d}{dt} g_{\bar k j} = g_{\bar k j} - R_{\bar k j} = \partial_j \partial_{\bar k} u$, where we normalize u by imposing the condition $\int_X e^{-u}\omega^n = \int_X \omega^n$. Here, $\omega = \frac{\sqrt{-1}}{2} g_{\bar k j} dz^j \wedge d\bar z^k \in \pi c_1(X)$ is the Kähler form of $g(t)$.

In the following, we will make use of the estimates of Perelman [P2] (see [ST]):

(i) For a uniform $C > 0$, we have $\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C$.
(ii) Let $\rho > 0$ be given. Then for all $x \in X$ and all r with $0 < r \leq \rho$ we have

$$\int_{B_r(x)} \omega^n > C' r^{2n}, \tag{2.1}$$

for a uniform constant $C' > 0$, where $B_r(x)$ is the geodesic ball of radius r centered at x with respect to $g(t)$.
(iii) The diameter of $(X, g(t))$ is uniformly bounded.

Define two time dependent inner products on $T^{1,0}$ by

$$\langle V, W\rangle_u = \int_X g_{\bar{k}j} V^j \overline{W^k} e^{-u} \omega^n \quad \text{and} \quad \langle V, W\rangle_0 = \int_X g_{\bar{k}j} V^j \overline{W^k} \omega^n. \tag{2.2}$$

Since u is uniformly bounded the corresponding norms $\|\cdot\|_u$ and $\|\cdot\|_0$ are equivalent. Let $\tilde{\lambda} = \tilde{\lambda}(t)$ and $\lambda = \lambda(t)$ respectively be the smallest positive eigenvalues of the operators $\tilde{L} = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}} + g^{i\bar{j}}\nabla_i u \nabla_{\bar{j}}$ and $L = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}$ acting on $T^{1,0}$ vector fields. Denote by $\mathrm{H}^0(X, T^{1,0})$ the space of holomorphic vector fields on X. Then $\tilde{\lambda}$ is the largest number satisfying

$$\int_X \left|\nabla_{\bar{i}} V^k\right|^2 e^{-u}\omega^n \geq \tilde{\lambda} \int_X |V^k|^2 e^{-u} \omega^n \tag{2.3}$$

for all V with the property: $\langle V, \xi\rangle_u = 0$ for all $\xi \in \mathrm{H}^0(X, T^{1,0})$. Similarly, λ is the largest number satisfying

$$\int_X \left|\nabla_{\bar{i}} V^k\right|^2 \omega^n \geq \lambda \int_X |V^k|^2 \omega^n \tag{2.4}$$

for all V with the property: $\langle V, \xi\rangle_0 = 0$ for all $\xi \in \mathrm{H}^0(X, T^{1,0})$. The following lemma shows that $\tilde{\lambda}$ and λ are uniformly equivalent.

Lemma 1 *There exist uniform positive constants* A_1 *and* A_2 *such that*

$$A_1\tilde{\lambda} \leq \lambda \leq A_2\tilde{\lambda}. \tag{2.5}$$

Proof of Lemma 1: Let $V \in T^{1,0}$ be a smooth vector field such that $\langle V, \xi\rangle_0 = 0$ for all $\xi \in \mathrm{H}^0(X, T^{1,0})$. Write

$$\begin{aligned} V = W + \xi_0 \quad &\text{with} \;\; \xi_0 \in \mathrm{H}^0(X, T^{1,0}) \\ \text{and} \quad \langle W, \xi\rangle_u = 0 \quad &\text{for all} \;\; \xi \in \mathrm{H}^0(X, T^{1,0}). \end{aligned} \tag{2.6}$$

Then

$$0 = \langle V, \xi_0\rangle_0 = \langle W, \xi_0\rangle_0 + \langle \xi_0, \xi_0\rangle_0, \tag{2.7}$$

and the Cauchy–Schwarz inequality implies

$$\langle \xi_0, \xi_0 \rangle_0^2 \leq \langle W, W \rangle_0 \langle \xi_0, \xi_0 \rangle_0. \tag{2.8}$$

Hence there exist $c_1, c_2 > 0$ such that

$$c_1 \langle \xi_0, \xi_0 \rangle_u \leq \langle \xi_0, \xi_0 \rangle_0 \leq \langle W, W \rangle_0 \leq c_2 \langle W, W \rangle_u. \tag{2.9}$$

Thus

$$\begin{aligned} \int_X |\bar{\nabla} V|^2 \omega^n &\geq c_3 \int_X |\bar{\nabla} V|^2 e^{-u} \omega^n = c_3 \int |\bar{\nabla} W|^2 e^{-u} \omega^n \\ &\geq c_3 \tilde{\lambda} \int_X |W|^2 e^{-u} \omega^n = \frac{c_3 \tilde{\lambda}}{2} \langle W, W \rangle_u + \frac{c_3 \tilde{\lambda}}{2} \langle W, W \rangle_u \\ &\geq \frac{c_3 \tilde{\lambda}}{2} \langle W, W \rangle_u + \frac{c_3 c_1 \tilde{\lambda}}{2 c_2} \langle \xi_0, \xi_0 \rangle_u \geq c_4 \tilde{\lambda} \langle V, V \rangle_0, \end{aligned} \tag{2.10}$$

and it follows that $\lambda \geq c_4 \tilde{\lambda}$, giving the first inequality. The second inequality follows similarly. □

We recall some notions of positivity. A tensor $T_{\bar{j}i\bar{l}k}$ is *Griffiths* nonnegative if

$$T_{\bar{j}i\bar{l}k} \overline{V^j} V^i \overline{W^l} W^k \geq 0 \tag{2.11}$$

for all vectors $V, W \in T^{1,0}$. For brevity we write $T_{\bar{j}i\bar{l}k} \geq_{Gr} 0$. The condition of nonnegative bisectional curvature means $R_{\bar{j}i\bar{l}k} \geq_{Gr} 0$. We say that a tensor $T_{\bar{j}i\bar{l}k}$ is *Nakano* nonnegative if

$$T_{\bar{j}i\bar{l}k} \overline{\zeta^{jl}} \zeta^{ik} \geq 0, \tag{2.12}$$

for all tensors $\zeta \in T^{1,0} \otimes T^{1,0}$, and we write $T_{\bar{j}i\bar{l}k} \geq_{Na} 0$ for short.

Next, we show that under a positive curvature condition, the eigenvalue λ can be bounded below away from zero.

Lemma 2 *Suppose that a Kähler metric g satisfies*

$$R_{\bar{j}i\bar{l}k} + R_{\bar{j}i} g_{\bar{l}k} - c\, g_{\bar{j}i} g_{\bar{l}k} \geq_{Na} 0, \tag{2.13}$$

for some constant $c > 0$. Then $\lambda \geq c$.

Proof of Lemma 2: Recall the commutation formulae:

$$(\nabla_i \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_i) V^k = g^{k\bar{m}} R_{\bar{l}i\bar{m}p} V^p \tag{2.14}$$

$$(\nabla_i \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_i) a_{\bar{j}} = g^{m\bar{q}} R_{\bar{l}i\bar{j}m} a_{\bar{q}}, \tag{2.15}$$

for a $T^{1,0}$ vector field V and a $(0, 1)$ form a. Let V be an eigenvector of the operator L with eigenvalue λ. Then

$$-g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} V^k = \lambda V^k. \tag{2.16}$$

Apply $\nabla_{\bar{l}}$ to obtain

$$-g^{i\bar{j}}\nabla_{\bar{l}}\nabla_i\nabla_{\bar{j}}V^k = \lambda\nabla_{\bar{l}}V^k. \tag{2.17}$$

Using the commutation formulae we have

$$-g^{i\bar{j}}\nabla_i\nabla_{\bar{l}}\nabla_{\bar{j}}V^k + g^{i\bar{j}}g^{k\bar{m}}R_{\bar{l}i\bar{m}p}\nabla_{\bar{j}}V^p + g^{i\bar{j}}g^{m\bar{q}}R_{\bar{l}i\bar{j}m}\nabla_{\bar{q}}V^k = \lambda\nabla_{\bar{l}}V^k. \tag{2.18}$$

Multiply by $g^{r\bar{l}}g_{\bar{t}k}\nabla_r\overline{V^t}$ to obtain

$$\begin{aligned} -g^{r\bar{l}}g_{\bar{t}k}g^{i\bar{j}}\nabla_r\overline{V^t}\nabla_i\nabla_{\bar{l}}\nabla_{\bar{j}}V^k &+ g^{r\bar{l}}g^{i\bar{j}}R_{\bar{l}i\bar{t}p}\nabla_r\overline{V^t}\nabla_{\bar{j}}V^p \\ &+ g^{r\bar{l}}g_{\bar{t}k}g^{m\bar{q}}R_{\bar{l}m}\nabla_r\overline{V^t}\nabla_{\bar{q}}V^k = \lambda g^{r\bar{l}}g_{\bar{t}k}\nabla_r\overline{V^t}\nabla_{\bar{l}}V^k. \end{aligned} \tag{2.19}$$

From (2.13), after integrating by parts:

$$\lambda\int_X \left|\nabla_{\bar{i}}V^k\right|^2\omega^n \geq c\int_X \left|\nabla_{\bar{i}}V^k\right|^2\omega^n + \int_X \left|\nabla_{\bar{i}}\nabla_{\bar{j}}V^k\right|^2\omega^n, \tag{2.20}$$

and hence $\lambda \geq c$. □

Next, we show, under a slightly different curvature assumption, that the eigenvalue $\tilde{\lambda}$ can be bounded below.

Lemma 3 *Suppose that a Kähler metric g satisfies*

$$R_{\bar{j}i\bar{l}k} + (1-c)g_{\bar{j}i}g_{\bar{l}k} \geq_{Na} 0, \tag{2.21}$$

for some constant $c > 0$. *Then* $\tilde{\lambda} \geq c$.

Proof of Lemma 3: Let V be an eigenvector of $\tilde{L}$ with eigenvalue $\tilde{\lambda}$. Then

$$-g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}V^k + g^{i\bar{j}}\nabla_{\bar{j}}V^k\nabla_i u = \tilde{\lambda}V^k. \tag{2.22}$$

Applying $\nabla_{\bar{l}}$ as before, using the commutation formulae and the definition of u we have

$$\begin{aligned} -g^{i\bar{j}}\nabla_i\nabla_{\bar{l}}\nabla_{\bar{j}}V^k &+ g^{i\bar{j}}g^{k\bar{m}}R_{\bar{l}i\bar{m}p}\nabla_{\bar{j}}V^p + g^{i\bar{j}}g^{m\bar{q}}R_{\bar{l}i\bar{j}m}\nabla_{\bar{q}}V^k \\ &+ g^{i\bar{j}}\nabla_{\bar{l}}\nabla_{\bar{j}}V^k\nabla_i u + \nabla_{\bar{l}}V^k - g^{i\bar{j}}R_{\bar{l}i}\nabla_{\bar{j}}V^k = \tilde{\lambda}\nabla_{\bar{l}}V^k. \end{aligned} \tag{2.23}$$

Multiply by $g^{r\bar{l}}g_{\bar{t}k}\nabla_r\overline{V^t}$ to obtain

$$\begin{aligned} -g^{r\bar{l}}g_{\bar{t}k}g^{i\bar{j}}\nabla_r\overline{V^t}\nabla_i\nabla_{\bar{l}}\nabla_{\bar{j}}V^k &+ (R_{\bar{j}i\bar{l}k} + g_{\bar{l}k}g_{\bar{j}i})\nabla^{\bar{j}}\overline{V^l}\nabla^i V^k \\ &+ g^{r\bar{l}}g_{\bar{t}k}g^{i\bar{j}}\nabla_r\overline{V^t}\nabla_{\bar{l}}\nabla_{\bar{j}}V^k\nabla_i u = \tilde{\lambda}g^{r\bar{l}}g_{\bar{t}k}\nabla_r\overline{V^t}\nabla_{\bar{l}}V^k. \end{aligned} \tag{2.24}$$

Integrating against $e^{-u}\omega^n$ we obtain

$$\tilde{\lambda}\int_X \left|\nabla_{\bar{i}}V^k\right|^2 e^{-u}\omega^n \geq c\int_X \left|\nabla_{\bar{i}}V^k\right|^2 e^{-u}\omega^n + \int_X \left|\nabla_{\bar{i}}\nabla_{\bar{j}}V^k\right|^2 e^{-u}\omega^n, \quad (2.25)$$

and hence $\tilde{\lambda} \geq c$. □

3 Proof of Theorem 1

For the proof of Theorem 1, we will need a number of lemmas.

Lemma 4 *Suppose the Mabuchi K-energy is bounded below on $\pi c_1(X)$ and the bisectional curvature of g_0 is nonnegative. Then along the Kähler–Ricci flow*

$$\|R_{\bar{k}j} - g_{\bar{k}j}\|_{C^0} \to 0,$$

as $t \to \infty$.

Proof of Lemma 4: By the results of Bando [B] and Mok [Mok], the non-negativity of the bisectional curvature is preserved along the Kähler–Ricci flow. Since the scalar curvature is uniformly bounded by Perelman's estimates, it follows that the bisectional curvatures, and hence the full curvature tensor of $g = g(t)$ is uniformly bounded along the flow. The covariant derivatives of the curvature are also uniformly bounded along the flow [S]. From [PS3], the lower boundedness of the Mabuchi K-energy implies

$$\int_X |R_{\bar{k}j} - g_{\bar{k}j}|^2\omega^n = \int_X |R-n|^2\omega^n \to 0, \quad (3.1)$$

as $t \to \infty$. Assume for a contradiction that there is a sequence of points x_i and times $t_i \to \infty$ with $|R_{\bar{k}j} - g_{\bar{k}j}|(x_i, t_i) \geq \varepsilon > 0$. Then by Perelman's non-collapsing result and the bound on the derivative of the Ricci curvature we obtain for uniform constants $r > 0$ and $c > 0$,

$$\int_{B_r(x_i)} |R_{\bar{k}j} - g_{\bar{k}j}|^2\omega^n \geq cr^{2n}, \quad (3.2)$$

at each time t_i. This contradicts (3.1). □

We will use the following result from [Che, Theorem 1.5], which is proved using the maximum principle argument of Mok [Mok].

Lemma 5 *Suppose there exist constants $c_0 > 0$ and $\nu > 1/2$ such that the following holds. There is a Kähler metric g_0 satisfying*

$$R_{\bar{j}i\bar{l}k}(g_0) - c_0((g_0)_{\bar{j}i}(g_0)_{\bar{l}k} + (g_0)_{\bar{j}k}(g_0)_{\bar{l}i}) \geq_{Gr} 0, \quad (3.3)$$

and the solution of the Kähler–Ricci flow $g = g(t)$ starting at g_0 satisfies

$$R_{\bar{j}i} \geq \nu g_{\bar{j}i}, \quad (3.4)$$

at all times. Then, along the Kähler–Ricci flow, $g = g(t)$ satisfies

$$R_{\bar{j}i\bar{l}k} - c_t(g_{\bar{j}i}g_{\bar{l}k} + g_{\bar{j}k}g_{\bar{l}i}) \geq_{Gr} 0, \tag{3.5}$$

for $c_t > 0$ with $\lim_{t\to\infty} c_t = (2\nu - 1)/(n+1) > 0$.

Proof of Lemma 5: For the reader's convenience, we give a sketch of the proof. Let c_t be a smooth positive function of t, equal to c_0 at $t = 0$, and to be determined later. Define a tensor $S_{\bar{j}i\bar{l}k}$ by

$$S_{\bar{j}i\bar{l}k} = R_{\bar{j}i\bar{l}k} - c_t(g_{\bar{j}i}g_{\bar{l}k} + g_{\bar{j}k}g_{\bar{l}i}). \tag{3.6}$$

The evolution of $S_{\bar{j}i\bar{l}k}$ along the Kähler–Ricci flow is given by

$$\begin{aligned}\frac{\partial}{\partial t}S_{\bar{j}i\bar{l}k} &= \Box S_{\bar{j}i\bar{l}k} + c_t(R_{\bar{j}i}g_{\bar{l}k} + R_{\bar{l}k}g_{\bar{j}i} + R_{\bar{j}k}g_{\bar{l}i} + R_{\bar{l}i}g_{\bar{j}k})\\ &\quad - \big(c_t(1+(n+1)c_t) + c_t'\big)(g_{\bar{j}i}g_{\bar{l}k} + g_{\bar{j}k}g_{\bar{l}i}),\end{aligned} \tag{3.7}$$

where, computing in an orthonormal frame for g, we define

$$\begin{aligned}\Box S_{\bar{j}i\bar{l}k} &= \frac{1}{2}(\nabla_p\nabla_{\bar{p}} + \nabla_{\bar{p}}\nabla_p)S_{\bar{j}i\bar{l}k} + S_{\bar{j}i\bar{l}k} + S_{\bar{j}i\bar{q}p}S_{\bar{p}q\bar{l}k} - S_{\bar{p}i\bar{q}k}S_{\bar{j}p\bar{l}q}\\ &\quad + S_{\bar{l}i\bar{q}p}S_{\bar{p}q\bar{j}k} - \frac{1}{2}(R_{\bar{p}i}S_{\bar{j}p\bar{l}k} + R_{\bar{j}p}S_{\bar{p}i\bar{l}k} + R_{\bar{p}k}S_{\bar{j}i\bar{l}p} + R_{\bar{l}p}S_{\bar{j}i\bar{p}k}).\end{aligned} \tag{3.8}$$

This follows readily from the evolution of the curvature tensor (see (2.3) of [PS2], for example) which, in the above notation, is given by

$$\frac{\partial}{\partial t}R_{\bar{j}i\bar{l}k} = \Box R_{\bar{j}i\bar{l}k}. \tag{3.9}$$

From (3.4) we have

$$\frac{\partial}{\partial t}S_{\bar{j}i\bar{l}k} \geq_{Gr} \Box S_{\bar{j}i\bar{l}k} - \big(c_t(1-2\nu+(n+1)c_t) + c_t'\big)(g_{\bar{j}i}g_{\bar{l}k} + g_{\bar{j}k}g_{\bar{l}i}). \tag{3.10}$$

We may assume, without loss of generality, that $(n+1)c_0 < 2\nu - 1$. Choose

$$c_t = \frac{(2\nu-1)\gamma e^{(2\nu-1)t}}{(n+1)(1+\gamma e^{(2\nu-1)t})}, \quad \text{for } \gamma = \frac{(n+1)c_0}{2\nu - 1 - (n+1)c_0} > 0, \tag{3.11}$$

so that $c_t(1-2\nu+(n+1)c_t) + c_t' = 0$. Then

$$\frac{\partial}{\partial t}S_{\bar{j}i\bar{l}k} \geq_{Gr} \Box S_{\bar{j}i\bar{l}k}. \tag{3.12}$$

We now apply the maximum principle argument of [Mok] which says that if a tensor $S_{\bar{j}i\bar{l}k}$ satisfies the same symmetry properties as the Riemann curvature tensor and also satisfies (3.12), then $S_{\bar{j}i\bar{l}k} \geq_{Gr} 0$ at $t = 0$ implies that $S_{\bar{j}i\bar{l}k} \geq_{Gr} 0$ for all $t > 0$. This completes the proof. □

We will also need the following lemma:

Lemma 6 *Suppose that the curvature of a Kähler metric g satisfies*

$$R_{\bar{j}i\bar{l}k} - cg_{\bar{j}i}g_{\bar{l}k} \geq_{Gr} 0, \tag{3.13}$$

for some constant $c > 0$. Then

$$R_{\bar{j}i\bar{l}k} + R_{\bar{j}i}g_{\bar{l}k} - ncg_{\bar{j}i}g_{\bar{l}k} \geq_{Na} 0. \tag{3.14}$$

Proof of Lemma 6: This result is an application of the argument of [D, Proposition 10.14]. It requires Lemma 10.15 from [D]:

Lemma 7 *Let $q \geq 3$ be an integer and let x^λ, y^λ for $1 \leq \lambda \leq n$ be complex numbers. Let U_q^n be the set of n-tuples of qth roots of unity and define complex numbers*

$$x'_{(\sigma)} = \sum_{\lambda=1}^{n} x^\lambda \overline{\sigma_\lambda}, \quad y'_{(\sigma)} = \sum_{\lambda=1}^{n} y^\lambda \overline{\sigma_\lambda}, \quad \textit{for each } \sigma = (\sigma_1, \ldots, \sigma_n) \in U_q^n.$$

Then for every pair (α, β) with $1 \leq \alpha, \beta \leq n$, the following holds:

$$q^{-n} \sum_{\sigma \in U_q^n} x'_{(\sigma)} \overline{y'_{(\sigma)}} \sigma_\alpha \overline{\sigma_\beta} = \begin{cases} x^\alpha \overline{y^\beta}, & \textit{if } \alpha \neq \beta \\ \sum_{\lambda=1}^n x^\lambda \overline{y^\lambda}, & \textit{if } \alpha = \beta. \end{cases} \tag{3.15}$$

Proof of Lemma 7: Although this lemma is already contained in [D], we give the short proof here for the sake of completeness. We only require the following elementary claim: the coefficient of $x^\lambda \overline{y^\mu}$ in the left hand side of (3.15) is $q^{-n} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\lambda} \sigma_\mu$, and this is equal to 1 if $\{\alpha, \mu\} = \{\beta, \lambda\}$ and 0 otherwise. Indeed, for the second alternative, assume without loss of generality that $\alpha \notin \{\beta, \lambda\}$ and then observe that

$$\sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\lambda} \sigma_\mu = \begin{cases} e^{2\pi i/q} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\lambda} \sigma_\mu, & \alpha \neq \mu \\ e^{4\pi i/q} \sum_{\sigma \in U_q^n} \sigma_\alpha \overline{\sigma_\beta}\, \overline{\sigma_\lambda} \sigma_\mu, & \alpha = \mu. \end{cases} \tag{3.16}$$

For (3.16), replace σ by the element of U_q^n obtained by multiplying the α component of σ by $e^{2\pi i/q}$. □

We may assume without loss of generality that we are calculating at a point where $g_{\bar{j}i} = \delta_{ji}$. Fix $\zeta \in T^{1,0} \otimes T^{1,0}$. We need to show

$$(R_{\bar{j}i\bar{l}k} + R_{\bar{j}i}g_{\bar{l}k} - ncg_{\bar{j}i}g_{\bar{l}k})\overline{\zeta^{jl}}\zeta^{ik} \geq 0. \tag{3.17}$$

Fix any integer $q \geq 3$ and for each σ in U_q^n let $V_{(\sigma)} = V_{(\sigma)}^i \partial/\partial z^i$ be the vector with components $V_{(\sigma)}^i = \sum_{\lambda=1}^n \zeta^{i\lambda} \overline{\sigma_\lambda} \in \mathbf{C}$. Let $W_{(\sigma)} = W_{(\sigma)}^k \partial/\partial z^k$ be the vector with components $W^k_{(\sigma)} = \sigma_k \in \mathbf{C}$. Then, by assumption,

$$\begin{aligned}0 &\leq \sum_{i,j,k,l} (R_{\bar{j}i\bar{l}k} - cg_{\bar{j}i}g_{\bar{l}k})q^{-n} \sum_{\sigma \in U_q^n} \overline{V^j_{(\sigma)}} V^i_{(\sigma)} \overline{W^l_{(\sigma)}} W^k_{(\sigma)} \\ &= \sum_{i,j} \sum_{k \neq l} R_{\bar{j}i\bar{l}k} q^{-n} \sum_{\sigma \in U_q^n} \overline{V^j_{(\sigma)}} V^i_{(\sigma)} \overline{\sigma_l} \sigma_k \\ &\quad + \sum_{i,j} \sum_{k=l} (R_{\bar{j}i\bar{l}k} - cg_{\bar{j}i}g_{\bar{l}k})q^{-n} \sum_{\sigma \in U_q^n} \overline{V^j_{(\sigma)}} V^i_{(\sigma)} \overline{\sigma_l} \sigma_k \\ &= \sum_{i,j} \sum_{k \neq l} R_{\bar{j}i\bar{l}k} \overline{\zeta^{jl}} \zeta^{ik} + \sum_{i,j,k} (R_{\bar{j}i} - ncg_{\bar{j}i}) \overline{\zeta^{jk}} \zeta^{ik}, \end{aligned} \tag{3.18}$$

where we have made use of Lemma 7. Hence

$$\begin{aligned}&(R_{\bar{j}i\bar{l}k} + R_{\bar{j}i}g_{\bar{l}k} - ncg_{\bar{j}i}g_{\bar{l}k})\overline{\zeta^{jl}}\zeta^{ik} \\ &= \sum_k \sum_{i,j} R_{\bar{j}i\bar{k}k} \overline{\zeta^{jk}} \zeta^{ik} + \sum_{i,j} \sum_{k \neq l} R_{\bar{j}i\bar{l}k} \overline{\zeta^{jl}} \zeta^{ik} + \sum_{i,j,k} (R_{\bar{j}i} - ncg_{\bar{j}i}) \overline{\zeta^{jk}} \zeta^{ik} \\ &\geq 0, \end{aligned} \tag{3.19}$$

since the first term is nonnegative by the assumption. □

We can now prove Theorem 1.

Proof of Theorem 1: If the initial metric has nonnegative bisectional curvature which is positive at one point then the bisectional curvature along the flow immediately becomes positive everywhere [B,Mok]. From Lemma 4 we see that for some $T > 0$ and $\nu > 1/2$ we have $R_{\bar{j}i} \geq \nu g_{\bar{j}i}$ when $t \geq T$. Without loss of generality then, we may assume that for $t \geq 0$ the metric has positive bisectional curvature and $R_{\bar{j}i} \geq \nu g_{\bar{j}i}$. From Lemmas 5, 6 and 2 we see that the eigenvalue λ is uniformly bounded away from zero. Since the Mabuchi K-energy is bounded below it follows from Theorem 2 of [PSSW2] (or, since the curvature is bounded, the results of [PS3]) that the Kähler–Ricci flow converges exponentially fast to a Kähler–Einstein metric. □

4 Proof of Theorem 2

Before we give the proof of Theorem 2, we recall the definition of a Kähler–Ricci soliton. We say that a metric g with Kähler form $\omega \in \pi c_1(X)$ is a Kähler–Ricci soliton if

$$g_{\bar{k}j} - R_{\bar{k}j} = \partial_j \partial_{\bar{k}} u \tag{4.1}$$

for a smooth function u with $\bar{\nabla}\bar{\nabla}u = 0$, or in other words if $\nabla^j u$ is a holomorphic vector field. If g is a Kähler–Ricci soliton then $g(t) = \Psi(t)^* g$ is a solution to the Kähler–Ricci flow, where $\Psi(t)$ is the 1-parameter subgroup of holomorphic automorphisms generated by the vector field $\mathrm{Re}(\nabla^j u)$. Sometimes, by abuse of notation, we also call $g(t)$ a Kähler–Ricci soliton.

Now we recall from [PS3] that for a solution $g(t)$ of the Kähler–Ricci flow, the function $Y(t) = \int_X |\nabla u|^2 \omega^n$ satisfies

$$\dot{Y}(t) \leq -2\lambda(t)Y(t) - 2\lambda(t)\,\mathrm{Fut}(\pi_t(\nabla^j u)) - Z(t), \tag{4.2}$$

where

$$Z(t) = \int_X |\nabla u|^2 (R - n) + \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n, \tag{4.3}$$

and $\mathrm{Fut}(\pi_t(\nabla^j u))$ is the Futaki invariant of the orthogonal projection π_t with respect to $\langle\, , \rangle_0$ of the vector field $\nabla^j u$ to the space $\mathrm{H}^0(X, T^{1,0})$ of holomorphic vector fields. We note that for Theorem 2, the Futaki invariant vanishes by hypothesis.

We have the following lemma.

Lemma 8 *If $g(t)$ is a Kähler–Ricci soliton then $\dot{Y}(t) = Z(t) = 0$ for all $t \geq 0$.*

Proof of Lemma 8: Since Y is unchanged by automorphisms it follows that $\dot{Y}(t) = 0$. Compute

$$\begin{aligned} \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n &= -\int_X \nabla^j u \nabla^{\bar{k}} u (\partial_j \partial_{\bar{k}} u)\omega^n \\ &= \int_X \big(\nabla_j \nabla^j u\big)\big(\nabla^{\bar{k}} u \nabla_{\bar{k}} u\big)\omega^n \\ &= \int_X (n - R)|\nabla u|^2 \omega^n, \end{aligned} \tag{4.4}$$

and hence $Z(t) = 0$. □

We will make use of the following result.

Theorem 4 *Suppose Condition (A') holds and that along the Kähler–Ricci flow we have $Y(t) \to 0$ as $t \to \infty$ and $\lambda(t) \geq c$ for some uniform constant $c > 0$. Then the Kähler–Ricci flow converges exponentially fast in C^∞ to a Kähler–Einstein metric.*

Proof of Theorem 4: This follows from the arguments of Lemma 5 and Lemma 6 of [PSSW2]. Indeed, one can easily check that the argument of Lemma 5 of [PSSW2] shows that under the assumptions of Theorem 4, $\|R(t) - n\|_{C^0}$ converges exponentially fast to zero. Now apply Lemma 6 of [PSSW2] which states that if $\int_0^\infty \|R(t) - n\|_{C^0} dt < \infty$ then the Kähler–Ricci flow converges exponentially fast in C^∞ to a Kähler–Einstein metric. □

We can now give the proof of Theorem 2.

Proof of Theorem 2: It is shown in [PS3] that if the Riemann curvature tensor is uniformly bounded along the flow and condition (B) holds then there is a uniform lower bound of $\lambda(t)$ away from zero. If $Y(t) \to 0$ as $t \to \infty$ then the required result will follow immediately from Theorem 4. We assume for a contradiction that there is a constant $\varepsilon > 0$ and a sequence of times $t_j \to \infty$ such that $Y(t_j) \geq \varepsilon$ for all j.

Since we have uniformly bounded curvature, diameter and injectivity radius along the flow we can apply Hamilton's compactness theorem [H2] to obtain (after passing to a subsequence) diffeomorphisms $F_j : \tilde{X} \to X$ such that $F_j^* g(t_j + t)$ converges to a solution $\tilde{g}(t)$ of the Kähler–Ricci flow on $\tilde{X}$ which is the same manifold as X, but with possibly a different complex structure $\tilde{J}$ (see [PS3]). The convergence of the metrics and their derivatives is uniform on compact subsets of $\tilde{X} \times [0, \infty)$. Moreover, $\tilde{g}$ is a Kähler–Ricci soliton.

This last assertion follows from a theorem in [ST], but for our particular case, we can give here a direct argument for the convenience of the reader. Given a solution $g(t)$ of the Kähler–Ricci flow one can make a change of variable $t = -\log(1 - 2s)$ and define a new metric $h = h(s)$ by $h(s) = (1 - 2s) g(t(s))$. Then h satisfies, in real coordinates, $\frac{\partial}{\partial s} h_{ij} = -2R_{ij}$ for $s \in [0, 1/2)$. Perelman [P1] showed that the functional

$$\mu(h, \tau) = \inf \Big\{ (2\tau)^{-n} \int_X (2\tau(R + |\nabla f|^2) + f - 2n) e^{-f} \omega^n \,\Big|\, (2\tau)^{-n} \int_X e^{-f} \omega^n = \int_X \omega^n \Big\},$$

where the metric quantities are those of h, satisfies $\frac{d}{ds} \mu(h(s), 1/2 - s) \geq 0$. By Perelman's estimates for the scalar curvature and Ricci potential, μ is uniformly bounded from above. Since μ is invariant under diffeomorphisms, it follows that the solution of the Ricci flow $\tilde{h}(s)$ corresponding to the limit solution $\tilde{g}(t)$ has $\mu(\tilde{h}(s), 1/2 - s)$ constant in s. Hence (see for example [KL, Sect. 12]) $\tilde{h}$ satisfies $\tilde{R}_{ij} + \tilde{\nabla}_i \tilde{\nabla}_j f - \frac{1}{1-2s} \tilde{h}_{ij} = 0$ for some $f = f(s)$ and it follows that $\tilde{g}$ is a Kähler–Ricci soliton, as required.

Now from (4.2),

$$\dot{Y}(t_j + t) \leq -2\lambda Y(t_j + t) - Z(t_j + t). \tag{4.5}$$

Since $\lim_{j \to \infty} Y(t_j + t) = \tilde{Y}(t)$, $\lim_{j \to \infty} Z(t_j + t) = \tilde{Z}(t)$ uniformly for t in any compact interval (where $\tilde{Y}$, $\tilde{Z}$ are the functions Y, Z corresponding to the metric $\tilde{g}$), we have

$$\dot{\tilde{Y}}(t) \leq -2\lambda \tilde{Y}(t) - \tilde{Z}(t). \tag{4.6}$$

But Lemma 8 says that $\dot{\tilde{Y}}(t) = \tilde{Z}(t) = 0$ so we get $\tilde{Y}(t) = 0$. This contradicts the assumption that $Y(t_j) \geq \varepsilon$ for all j and completes the proof of Theorem 2. □

5 Proof of Theorem 3

We now consider the case when $n \leq 2$ and the Futaki invariant of X vanishes. The key lemma that makes use of complex dimension 1 or 2 is as follows:

Lemma 9 *Suppose X has complex dimension $n \leq 2$. Then*

$$R_{\bar{j}i\bar{l}k} \geq_{Gr} 0 \Longleftrightarrow R_{\bar{j}i\bar{l}k} \geq_{Na} 0.$$

Proof of Lemma 9: The case $n = 1$ is trivial. Assume $n = 2$ and that g has nonnegative bisectional curvature. We require $R_{\bar{j}i\bar{l}k}\overline{\zeta^{jl}}\zeta^{ik} \geq 0$ for all ζ. Note that we only need the inequality for symmetric ζ since if we set $\nu^{ik} = (\zeta^{ik} + \zeta^{ki})/2$ then by the symmetry of the curvature tensor,

$$R_{\bar{j}i\bar{l}k}\overline{\zeta^{jl}}\zeta^{ik} = R_{\bar{j}i\bar{l}k}\overline{\nu^{jl}}\nu^{ik}. \tag{5.1}$$

We assume then that ζ is symmetric and of rank 2 (if ζ has rank 1 the result follows easily). Make a linear change of complex coordinates so that ζ is the identity. Denote these new coordinates by z^1, z^2. Then

$$R_{\bar{j}i\bar{l}k}\overline{\zeta^{jl}}\zeta^{ik} = R_{\bar{1}1\bar{1}1} + R_{\bar{2}1\bar{2}1} + R_{\bar{1}2\bar{1}2} + R_{\bar{2}2\bar{2}2}. \tag{5.2}$$

We will show that the right hand side is nonnegative. Write $X = \partial/\partial z^1$ and $Y = \partial/\partial z^2$. Calculate

$$\begin{aligned} 0 &\leq R(\overline{X} - i\overline{Y}, X + iY, -i\overline{X} + \overline{Y}, iX + Y) \\ &= R(\overline{X}, X, \overline{X}, X) + R(\overline{Y}, X, \overline{Y}, X) + R(\overline{X}, Y, \overline{X}, Y) + R(\overline{Y}, Y, \overline{Y}, Y) \\ &= R_{\bar{1}1\bar{1}1} + R_{\bar{2}1\bar{2}1} + R_{\bar{1}2\bar{1}2} + R_{\bar{2}2\bar{2}2}, \end{aligned} \tag{5.3}$$

where to go from the first to the second line, we have cancelled some terms using the symmetry of the curvature tensor. □

Remark Note that positive bisectional curvature in dimension 2 is not equivalent to positive curvature in the Nakano sense. Indeed, a Kähler manifold with $n \geq 2$ can never have positive Nakano curvature because $R_{\bar{j}i\bar{l}k}\overline{\zeta^{jl}}\zeta^{ik} = 0$ for every skew-symmetric ζ.

Proof of Theorem 3: From Lemma 9 and Lemma 3 we see that $\tilde{\lambda}(t) \geq 1$ and so, by Lemma 1, $\lambda(t)$ is uniformly bounded below away from zero along the flow. We can now argue in the same way as in the proof of Theorem 2. □

Acknowledgements The authors wish to thank the referees for their careful reading of the manuscript and for some helpful suggestions.

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