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On the algebraic constructibility of varieties of integrable rational functions on $\mathbf{C}^n$ *

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1. Introduction

The degeneracy of holomorphic functions is a notion difficult to quantify in the case of several variables. In recent years, the following invariant has attracted considerable attention: Let $f(z)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbf{C}^n$ with $f(0) = 0$. Then the critical integrability exponent $\delta_f(0)$ of f at 0 is defined by

$$\delta_f(0) \;=\; \sup\left\{\delta : \int_U |f(z)|^{-\delta} dV < \infty \;\text{ for some } U\right\}$$

Here U denotes a polydisk centered at 0 and dV is the usual Euclidean measure. The invariant $\delta_f(0)$ plays an important role in Kähler geometry [S1][T1-2][TY][Y]. Its stability under perturbations has been established for dimension $n = 2$ in [T2] and [PS1], for one-dimensional holomorphic perturbations in [S2] and [PS1], and in full generality in [DK]. Analogous notions for real-analytic functions have been investigated in [K] and [PSS1]. Their role in the study of oscillatory integrals and oscillatory integral operators is described in [V] and [P][PS1][PS2][PSS2] respectively.

In this paper, we shall investigate the global integral:

$$I(f) \;=\; \int_{\mathbf{C}^n} |f(z)|^{-\delta} dV\ .$$

Unlike the local integrals described above, the finiteness of such global integrals is not stable under holomorphic perturbations. Examples of algebraic families of polynomials f_t with $I(f_0) < \infty$ but $I(f_t) = \infty$ for all $t \neq 0$ are given in

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Section VI. In particular, I is not a continuous function of f. On the other hand, we shall prove that I is continuous almost everywhere. Furthermore, the space of polynomials for which the global integral converges turns out to retain some important algebraic properties, namely that it is *algebraically constructible*.

Recall that a subset $\mathcal{F}$ of an algebraic variety X is *constructible* if it can be written as a finite union of sets of the form $V_i \cap W_i'$

$$\mathcal{F} = \cup_{i=1}^{N} V_i \cap W_i'$$

where V_i, $W_i \subseteq X$ are subvarieties and W_i' is the complement of W_i in X (see e.g. [Ha]). This notion arises naturally in algebraic geometry: For example, it is well known that if $f : X \to Y$ is a morphism of algebraic varieties, then while the image of f need not be open or closed, it is constructible. More generally, the image of any constructible subset of X is constructible. A second example is the basic theorem of Chevalley which states that for every non-negative integer d, the set $\{y \in Y : \dim(f^{-1}(y)) = d\}$ is constructible.

We now describe the algebraically contructible sets arising in our context as well as the continuity properties of the function I in the general setting of rational functions $R(z)$ rather than polynomials $f(z)$. Let X be the space of all non-zero rational functions on $\mathbf{C}^n$ of degree bounded by a fixed integer N, and let $\delta \in \mathbf{R}$. Since a rational function is determined by its coefficients, we may identify X with (the projective image of) an open subset of $\mathbf{C}^m$ for some m (depending on n and N). We will show that the subset of X consisting of those rational functions $R(z)$ with $|R(z)|^\delta \in L^1(\mathbf{C}^n)$ is an algebraically constructible set. In other words, there exist finite collections of polynomials $F_{i,j}$ and G_i in m variables such that a rational function $R(z)$ satisfies $|R(z)|^\delta \in L^1(\mathbf{C}^n)$ if and only there exists i with $G_i(R) \neq 0$ and $F_{i,j}(R) = 0$ for all j. Furthermore, the map $I : X \to (0, \infty]$ given by $I(R) = \int_{\mathbf{C}^n} |R(z)|^\delta$ is piecewise continuous in the sense that X can be partitioned into a finite disjoint union of algebraically constructible subsets, restricted to each of which the function I is continuous. In particular, I is continuous almost everywhere.

Thus, for example, when $p > 1$ we see that the set of rational functions of degree N which are in $L^p(\mathbf{C}^n)$ is a constructible subset of all rational functions of degree N. Moreover, the L^p norm of R is a piecewise continuous function of the coefficients of R.

Our method is based on three fundamental ingredients. The first is the “algebraic estimates" of [PS1], suitably extended to the present global context. The second is a uniform perturbation theorem, which shows, roughly speaking, that integrability is preserved *uniformly* in f when the exponent δ is perturbed. The proof of the perturbation theorem requires in turn the third fundamental ingredient, which is the Bierstone-Milman approach to resolution of singularities [BM88a]. The Bierstone-Milman approach is precise and powerful enough to let us control uniformly the degrees of the monomials in the resolution of singularities of a function f in terms of the degree of f. We discuss these issues now in greater detail.

For the purpose of our inductive arguments, we need to investigate more general integrands. These are the "absolute rational powers" introduced in [PS1]:

Fix positive integers k, l, n, p, real numbers $\epsilon, \delta, \sigma_1, ..., \sigma_n$, with $\epsilon, \delta \geq 0$ and, for $1 \leq i \leq k$ and $1 \leq j \leq l$, non-zero polynomials $P_i(b; z), Q_j(b; z) \in \mathbf{C}[b, z] = \mathbf{C}[b_1, ..., b_p; z_1, ..., z_n]$. Set

$$\mathcal{B} = \{b \in \mathbf{C}^p : P_i(b; z) \neq 0 \text{ for some } z \in \mathbf{C}^n \text{ and some } i, \ 1 \leq i \leq k\}.$$

For fixed data ξ

$$\xi = (\epsilon, \delta, \sigma_1, ..., \sigma_n) \in \mathbf{R}^{n+2},$$

and for each $b \in \mathcal{B}$, consider the function

$$R(P_i, Q_j; \xi; b; z) = R(\xi; b; z) = \frac{\sum_{i=1}^{k} |P_i(b; z)|^{\epsilon}}{\sum_{j=1}^{l} |Q_j(b; z)|^{\delta}} \cdot \prod_{t=1}^{n} |z_t|^{\sigma_t} . \qquad (1.1)$$

When $\sigma_i = 0$ for all i, we shall write

$$R(\xi; b; z) = R(\epsilon, \delta; b; z) = K(b; z)/L(b; z).$$

In this case, we say that R is an "absolute rational power" (ARP) in the variables $b_1, ..., b_p, z_1, ..., z_n$. The set of all such ARP's will be denoted by $\mathbf{C}|(b; z)|$. The numerator K, and the denominator, L, are called "absolute polynomial powers". Although we are mainly interested in ARP's, our argument involves changes of variables of the form $z_t \mapsto z_t^{-1}$. Such a change of variables does not leave the class of ARP's invariant, and thus it is convenient to introduce the somewhat larger class of functions defined by (1.1).

Let $R(\xi : b; z) = R(b; z)$ be as in (1.1) and let

$$I(b) = I(\xi, b) \ = \int_{\mathbf{C}^n} R(\xi, b; z) \, dV \qquad (1.2)$$

where dV is the standard euclidean measure on $\mathbf{C}^n$.

A subset $X \subseteq \mathbf{C}^p$ is called an algebraic variety if there exist a finite collection of polynomials $F_i \in \mathbf{C}[z_1, ..., z_p]$ such that $X = \{z \in \mathbf{C}^p : F_i(z) = 0 \text{ for all } i\}$. A subset $V \subseteq \mathcal{B}$ is called an algebraic sub-variety of $\mathcal{B}$ if $V = \mathcal{B} \cap X$ for some algebraic variety $X \subseteq \mathbf{C}^p$.

Our main result is the following:

Theorem 1. *Fix $\xi = (\epsilon, \delta, \sigma_1, ..., \sigma_n)$ as above, and define $I = I(\xi, b)$ as in (1.2). Then there exists a filtration by algebraic sub-varieties*

$$\mathcal{B} = V_0 \supset V_1 \supset \cdots V_{\mathcal{M}} = \emptyset$$

such that $I|_{V_\lambda \backslash V_{\lambda+1}}$ *is continuous for* $0 \leq \lambda \leq \mathcal{M} - 1$. *In particular,* $I(b)$ *is continuous almost everywhere. Moreover, if* $I(b) = \infty$ *for some* $b \in V_\lambda \backslash V_{\lambda+1}$ *then* $I(b) = \infty$ *for all* $b \in V_\lambda \backslash V_{\lambda+1}$.

Corollary. *The set*

$$\mathcal{F} = \mathcal{F}_\xi = \left\{ b \in \mathcal{B} : \int_{\mathbf{C}^n} R(\xi; b; z)\, dV_1 \cdots dV_n < \infty \right\}$$

is algebraically constructible.

Proof. Let $S = \{\lambda : I(b) = \infty$ for all $b \in V_\lambda \backslash V_{\lambda+1}\}$. Then $\mathcal{F} = \cup_{\lambda \in S} V_\lambda \backslash V_{\lambda+1}$.

We shall show in a forthcoming work how Theorem 1 and the Corollary can be applied to the study of the analyticity of the level sets of the critical integrability exponent $\delta_f(z)$.

We now outline the proof of Theorem 1. As mentioned earlier, a first important component of the proof is “global algebraic estimates". In view of [PS1], Theorem 4, we know that the integral of a rational expression in one variable over a ball of *finite* radius r is equal, in size, to a piecewise rational expression of the coefficients (hence the terminology “algebraic"). Theorem 4 of [PS1] requires also two conditions:

(a) the roots of the denominator have to be sufficiently close to the origin, and
(b) the exponent vector ξ has to be non-degenerate (c.f. (2.1) below).

In our context, we need global algebraic estimates, in the sense that the domain of integration is now $\mathbf{C}$. We show in Section II (see Theorem 2) that such estimates do hold. In the process, the first condition (a) is naturally eliminated. However, the second condition (b) on the non-degeneracy of the exponent vector ξ is still required.

Naively, iteration of the one-dimensional global result (using Fubini’s theorem) should prove the main theorem. The difficulty in implementing this iterative procedure arises from the non-degeneracy condition (b) on the exponent vector ξ: Even if we start with non-degenerate exponents, any integration may lead to rational expressions whose exponent vector is degenerate.

This is why the key uniform perturbation theorem (Theorem 3) is needed. This theorem states that the exponent vector ξ can be slightly perturbed to a non-degenerate exponent vector ξ', in such a way that

$$\{b \in \mathcal{B}; I(\xi, b) < \infty\} \subset \{b \in \mathcal{B}; I(\xi', b) < \infty\}.$$

By the usual theorem on resolution of singularities, it is easy to see that for $b \in \mathcal{B}$ fixed, there exists $\mu(b) > 0$ such that $I(b) < \infty$ implies $\tilde{I}(b) < \infty$ provided the distance between ξ and ξ' is less than $\mu(b)$. The perturbation theorem is similar, but with the crucial additional property that $\mu(b)$ *can be taken as a fixed number*

μ independent of b. Remarkably, this strengthening is valid if the multiplicities of the monomials in the resolution of an algebraic function $f(z)$ are known to be bounded uniformly in terms of the degree of $f(z)$. Assuming these uniform bounds on multiplicities, the strengthening is established in Section 3.

A resolution of singularities with uniform bounds for the multiplicities of the resulting monomials is a refinement of the usual theorem which may be termed “controlled resolution of singularities". To obtain this refinement, we invoke the final important component of the proof, namely the Bierstone-Milman approach to resolution of singularities.

The “controlled resolution of singularities" theorem has been established in [BM99], using the techniques of [BM88a],[BM88b], and [BM91]. In this paper we actually need only a weaker and more elementary version: “controlled uniformization of singularities". We show that the algorithm of [BM88a],[BM88b], and [BM89] yields this more elementary result: If we follow the steps of the Bierstone-Milman algorithm and keep track of the multiplicities of the local blow-ups, we obtain the desired bounds. As pointed out by the referee, controlled uniformization can also be obtained from the usual resolution of singularities by successive “simultaneous" resolutions of polynomials in a stratification of Zariski open sets. However, in order to reach a broader audience, we have presented a detailed and self-contained treatment following the Bierstone-Milman algorithm in Section V.

2. Global algebraic estimates

It may be helpful to motivate the main result of this section by an explicit example. Consider the one-dimensional global integral

$$I(b) = \int_{\mathbf{C}} \frac{1}{|b_0 z^3 + b_2 z + b_3|^\delta} dV(z), \quad b = (b_0, b_2, b_3).$$

where $1 < \delta < 2$ is a fixed exponent (the other ranges of δ are easier). Although $I(b)$ is a complicated function of the parameter b, one can show that its *size* $I(b)$ is very simple:

$$I(b) \sim \frac{1}{|b_0|^{1-\frac{\delta}{2}} (|b_2|^6 + |b_0 b_3^2|^2)^{\frac{1}{3}-\frac{\delta}{4}} |4b_2^3 + 27 b_0 b_3^2|^{\delta-1}}.$$

Here we have introduced the following notation which will be used systematically in the sequel. If $F(b)$ and $G(b)$ are two $(0, \infty]$ valued functions, then we write $F \sim G$ if there exists a constant $c > 0$ such that $cF(b) \leq G(b) \leq c^{-1}F(b)$. Here c, the “implied constant", is independent of b.

The key feature of the preceding inequality is that the right hand side is again essentially a rational expression, this time in the coefficients (b_0, b_2, b_3) of the integrand in $I(b)$. A local version of such estimates, i.e., a version where the domain of integration is a compact disk in $\mathbf{C}$, had been proved in [PS1]. Our main task in this section is to obtain the following global version.

Let $n = 1$ and fix $R(\xi; b; z)$ as in (1.1). Assume that $\sigma_i = 0$ for all i, so that $R(\xi; b; z) = R(\epsilon, \delta; b; z)$ is an absolute rational power. Recall that the notations $\mathbf{Z}|(b)|$, $K_\lambda(b)/L_\lambda(b)$ have been defined in the Introduction.

Theorem 2 (Global algebraic estimates). *Let M (resp. N) be the highest power of z appearing in the P_i (resp. Q_j) and assume that ϵ, δ are rational numbers which are non-degenerate in the sense that*

$$\mu\epsilon - \nu\left(\frac{\delta}{[\delta]+1}\right) + 2 \neq 0 \tag{2.1}$$

for all integers μ, ν such that $0 \leq \mu \leq M$ and $0 \leq \nu \leq N([\delta]+1)$. Then there exists a finite chain $\mathcal{U}_\lambda$, $0 \leq \lambda \leq \mathcal{N}$, of algebraic varieties

$$\mathcal{B} = U_0 \supset U_1 \supset \cdots \supset U_{\mathcal{N}} = \emptyset\ ;$$

and a corresponding sequence of absolute rational powers, $T_\lambda \in \mathbf{Z}|(b)|$, $0 \leq \lambda \leq \mathcal{N} - 1$, with the following properties:

1. *If $T_\lambda(b) = K_\lambda(b)/L_\lambda(b)$ and if $b \in U_\lambda \backslash U_{\lambda+1}$, then $K_\lambda(b) \neq 0$. In particular, T_λ is defined, and nowhere vanishing on $U_\lambda \backslash U_{\lambda+1}$.*
2. *We have*

$$\mathcal{U}_{\lambda+1} = \{b \in U_\lambda\ :\ K_\lambda(b) = 0\}\ =\ \{b \in U_\lambda\ :\ K_\lambda(b) = 0 \text{ and } L_\lambda(b) = 0\}$$

3. *The following estimate holds:*

$$\int_{\mathbf{C}} R(b; z)\, dV \sim T_\lambda(b) \tag{2.2}$$

for all $b \in U_\lambda \backslash U_{\lambda+1}$. If we let $\lambda(b)$ be the largest value of λ for which $b \in U_\lambda$, then we can write (2.2) as follows:

$$\int_{\mathbf{C}} R(b; z)\, dV \sim T_{\lambda(b)}(b) \tag{2.3}$$

The implied constants are independent of b.

Remark. We do allow for the possibility $L_\lambda(b) \equiv 0$, in which case $T_\lambda(b) \equiv \infty$ for $b \in U_\lambda \backslash U_{\lambda+1}$.

Proof of Theorem 2. Theorem 4 of [PS1] gives a local version of Theorem 2: Its statement can be obtained from that of Theorem 2 by replacing conclusion 3. with the statement:

3'. Let $N_i = deg(Q_i)$ and assume $N_1 \geq N_2 \geq \cdots N_l$. Choose the notation in such a way that $Q_j(b, z) = b_j z^{N_j} +$ lower order terms, $1 \leq j \leq l$. Let $B(r) = \{z \in \mathbf{C}; |z| < r\}$. Then there exists a number $s > 0$ with the following property:

For $b \in U_\lambda \backslash U_{\lambda+1}$ satisfying $|||\ b_1^{-1} Q_1(b; Z) - Z^{N_1}||| < s$ and $|||\ b_1^{-1} Q_j(b; Z)||| < 2$ for $j > 1$, we have

$$\int_{B(1)} R(b; z)\, dV \sim T_\lambda(b) \tag{2.4}$$

Here we made use of the notation

$$\left|\left|\left| \sum_{\mu=0}^{M} b_\mu z^\mu \right|\right|\right| = \sup_{0 \leq \mu \leq M} |b_\mu|.$$

Returning to the proof of Theorem 2, we observe that

$$I(b) = \int_{\mathbf{C}} R(b; z)\, dV = \lim_{r \to \infty} \int_{B(r)} R(b; z)\, dV = \lim_{r \to \infty} r^2 \int_{B(1)} R(b; rz)\, dV$$

Let $M_i = deg(P_i)$ and assume $M_1 \geq \cdots \geq M_k$. Then, letting $t = 1/r$ we obtain:

$$I(b) = \lim_{t \to 0} t^{N_1 \delta - M_1 \epsilon - 2} \int_{B(1)} R(\beta_b(t); z)\, dV \tag{2.5}$$

where $\beta(t) = \beta_b(t) = (b_1 t^{a_1}, ..., b_p t^{a_p})$ for some non-negative integers a_1. Thus, for a fixed b, $\beta_b(t)$ is an algebraic curve in $\mathcal{B}$. Moreover, $R(\beta_b(1); z) = R(b; z)$.

At this point we would like to apply (2.4) in order to evaluate the size of the integral appearing in (2.6). We cannot do this directly since the condition $|||b_1^{-1} Q_j(\beta(t); Z)||| < 2$ may be violated: For t small, if $N_j = N_1$, $|||b_1^{-1} Q_j(\beta(t); Z)|||$ approaches $|b_j/b_1|$ which may be larger than 2. Of course, some $|b_j|$ must be greater than or equal to all the others, so it is tempting to assume that $|b_1|$ is the largest. But the condition $|b_1| \geq |b_i|$ for all i is *not* an algebraic condition, and the subset of $\mathcal{B}$ which it defines is thus not a constructible set.

To circumvent this difficulty, we make use of the following result of [PS1], which essentially reduces the case of several polynomials $Q_1(b, z), \cdots, Q_l(b, z)$ in the denominator of $R(b, z)$ to the case of a single polynomial. According to Lemma 4.9 of [PS1], there exists $D > 0$ such that if

$$|||b_1^{-1} Q_1(b; Z) - Z^{N_1}||| < s/(2l) \text{ and } |||b_1^{-1} Q_j(b; Z)||| < s/(2l)$$

for $j > 1$, then

$$\int_{B(1)} R(b; z)\, dV \sim \inf_{\{\theta \in (\mathbf{R}/Z)^l; D\theta = 0\}} \int_{B(1)} \frac{\sum_{i=1}^{k} |P_i(b; z)|^\epsilon}{|\sum_{j=1}^{l} Q_j(b; z) e(\theta_j)|^\delta}\, dV\ , \tag{2.6}$$

where $\theta = (\theta_1, ..., \theta_l) \in (\mathbf{R}/Z)^l$ and $e(x) = e^{2\pi i x}$. We would like to use (2.6) in order to reduce to the case where $l = 1$. Of course the hypothesis for (2.6) is very similar to that of (2.4), and thus appears to present the same difficulty - that of not knowing, in an algebraic way, which $|b_i|$ is maximal. However, we can handle this problem by considering all the b_i at once, as follows: Choosing s as in 3' we have, for all $t > 0$,

$$\int_{B(1)} R(\beta(t); z)\, dV \sim \inf_{\{\mu:1\leq\mu\leq l\}} \int_{B(1)} \frac{\sum_{i=1}^{k} |P_i(\beta(t), z)|^{\varepsilon}}{|Q_\mu(\beta(t), z)|^{\delta} + (s/3l) \sum_{j\neq\mu} |Q_j(\beta(t), z)|^{\delta}}$$

$$\leq \inf_{\{1\leq\mu\leq r\}} \inf_{\{\theta\in(\mathbf{R}/\mathbf{Z})^l; D\theta=0\}} \int_{B(1)} \frac{\sum_{i=1}^{k} |P_i(\beta(t); z)|^{\epsilon}}{|Q_\mu(\beta(t); z)e(\theta_\mu) + \frac{s}{3l} \sum_{j\neq\mu}^{l} Q_j(\beta(t); z)e(\theta_j)|^{\delta}}\, dV \tag{2.7}$$

For t sufficiently small, the inequality of (2.7) becomes an equivalence ($\sim$): Assume that $|b_1| \geq |b_i|$ for $1 \leq i \leq r$. Then $Q_1/b_1 - z^{N_1} = O(t)$ and $|(s/3l)Q_j/b_1| = (s/3l)|(b_j/b_1 + O(t))|$ which, for t sufficiently close to zero, is smaller than $s/2l$. Thus (2.6) implies that (2.7) becomes an equivalence for t sufficiently small. Moreover, (2.4) implies that for t sufficiently small, the sizes of all the integrals on the right side of (2.7) are equal to certain absolute rational powers $T_\lambda(b, t; \mu, \theta)$.

Finally we multiply both sides of (2.7) by $t^{N_1\delta - M_1\epsilon - 2}$ and take the limit as $t \to 0$. Arguing as in (4.26)-(4.30) of [PS1], we see that the limit exists and equals $\inf_{\mu,\theta} T_\lambda(b; \mu, \theta)$ for certain $T_\lambda(b, \mu, \theta)$. Finally, setting $T_\lambda(b) = (\sum_{\mu,\theta} T_\lambda(b; \mu, \theta)^{-1})^{-1}$, we clearly have the equivalence $T_\lambda(b) \sim \inf_\mu T_\lambda(b; \mu, \theta)$ which, when combined with (2.5), yields the theorem.

3. The uniform perturbation theorem

Theorem 3. *Fix* $R(\xi) = R(\xi, b, z)$ *as in* (1.1). *Then there exists* $\mu > 0$ *such that for each* $b \in \mathcal{B}$,

$$\int_{\mathbf{C}^n} R(\xi; b; z)\, dV_1 \cdots dV_n < \infty \quad \text{implies} \quad \int_{\mathbf{C}^n} R(\xi'; b; z)\, dV_1 \cdots dV_n < \infty$$

for all ξ' *such that* $|\xi' - \xi| < \mu$. *Here* μ *depends on the fixed data* ξ, *but is independent of* b.

The function $R(\xi; b, z)$ remains unchanged if we replace δ by $\delta/([\delta] + 1)$ and Q_j by $Q_j^{[\delta]+1}$. Thus we may assume in the proof of Theorem 3, that $\delta < 1$. As explained in the Introduction, the proof of Theorem 3 requires the following theorem on “controlled resolution of singularities".

Let M be a complex manifold. We shall say that a family of holomorphic mappings of complex manifolds, $\{\pi_j : M_j \to M\}$, is a locally finite covering of M if:

(1) The images $\pi_j(M_j)$ are subordinate to a locally finite covering of M by open subsets (that is, a covering $M = \cup\, U_\alpha$ by open subsets with the property that every compact subset of M has non-empty intersection with only a finite number the U_α. To say that the $\pi_j(M_j)$ are subordinate to the covering means that for every j, there exists an α such that $\pi_j(M_j) \subseteq U_\alpha$).
(2) If K is a compact subset of M, then there are compact subsets K_j of M_j such that $K = \cup \pi_j(K_j)$ (the union is finite by (1)).

Theorem 4. (Controlled uniformization) *Fix an integer* $n > 0$. *There is a function* $D : \mathbf{N} \to \mathbf{N}$ *with the following property: For every open set* $M \subseteq \mathbf{C}^n$, *and every polynomial function* $f : M \to \mathbf{C}$, *there exists a locally finite covering* $\{\pi_j : B_j \to M\}$, *of* M,*with* B_j *a polydisk in* $\mathbf{C}^n$, *such that*

(1) For each j, $f \circ \pi_j(x) = u_j(x) m_j(x)$ *where* $u_j(x)$ *is holomorphic,* $|u_j(x)| \geq c$ *for some* $c > 0$, *and* $m_j(x)$ *is a monomial.*
(2) $\sup_j \deg(m_j(x)) \leq D(\deg(f))$.
(3) If $Jac(\pi_j)$ *denotes the jacobian determinant, then* $Jac(\pi_j) = v_j(x) n_j(x)$ *where* $n_j(x)$ *is a monomial,* $\sup_j \deg(n_j(x)) \leq D(\deg(f))$ *and* $v_j(x)$ *is holomorphic,* $|v_j(x)| \geq c$ *for some* $c > 0$.
(4) The map $\pi_j : B_j \backslash \pi_j^{-1} f^{-1}(0) \to M \backslash f^{-1}(0)$ *is a local isomorphism.*

Parts (2) and (3) of Theorem 4 are the key additional ingredients which we need, in order to show that the perturbation μ of Theorem 3 can be chosen independent of the parameter b. Part (1) of Theorem 4 follows easily from Theorem 3.1 of [BM88b]. In fact, the proof of Theorem 3.1 in [BM88b] contains an algorithm for constructing the local covering: If one traces through (a slightly modified version of) that algorithm, and keeps track of the exponents which appear in the blowings-up, one obtains (2) and (3) as well. For the sake of completeness, we shall provide the necessary details in Section V.

For now, we assume Theorem 4 and proceed with the proof of Theorem 3.

Proof of Theorem 3. Fix $D' \subseteq D \subseteq \mathbf{C}^n$ bounded open sets with $\overline{D}' \subseteq D$, where $\overline{D}'$ is the closure of D'. We first prove the following:

Claim. There exists $\mu > 0$ such that for $b \in \mathcal{B}$,

$$\int_D R(\xi; b; z)\, dV_1 \cdots dV_n < \infty \quad \text{implies} \quad \int_{D'} R(\xi' : b; z)\, dV_1 \cdots dV_n < \infty \tag{3.1}$$

for all ξ' such that $|\xi' - \xi| < \mu$.

Proof of claim. Recall that the function $R(\xi, b, z)$ is given by the expression (1.1). Without loss of generality we may assume that $k = 1$. We show next that we may assume that $l = 1$ as well: Let

$$C \;=\; \{(w_1, ..., w_l) \in \mathbf{C}^l;\, 1 \leq |w_j| \leq 2 \;\text{ for all }\; j\}$$

Then, making use of Lemma 4.4 of [PS1], we have

$$\int_D R(\xi, b, z)\, dV \sim \int_{C\times D} \frac{|P(b,z)|^\epsilon}{|\sum_{j=1}^l w_j Q_j(b,z)|^\delta} \prod_{i=1}^n |z_i|^{\sigma_i}\, dWdV$$
$$= \int_{C\times D} R(\xi, b, w, z) dWdV$$

where $dV = dV_1 \cdots dV_n$ and $dW = dW_1 \cdots dW_l$ are the euclidean measures (note that Lemma 4.4 applies since we are assuming that $\delta < 1$). The function $R(\xi, b, w, z)$ is a rational expression in $l + n$ variables, but it has only one term in the denominator. This shows that we may assume, in the proof of the claim, that $k = l = 1$.

Thus $R = |P|^\epsilon |Q|^{-\delta} \prod_k |z_k|^{\sigma_k}$. Let $f = PQ\prod_k z_k$. Using Theorem 4, we choose a locally finite cover $\pi_i : B_i \to D$ such that $f \circ \pi_i$ is a monomial of uniformly bounded degree times a nowhere vanishing factor. Since $\overline{D}'$ is compact there is an integer m such that $\overline{D}'$ has empty intersection with $\pi_i(B_i)$ for all $i > m$. For $i \leq m$,

$$\int_D R(\xi) < \infty \Longrightarrow \int_{\pi_i(B_i)} R(\xi) < \infty \Longrightarrow \int_{B_i} (R(\xi)\circ\pi_i) J(\pi_i, b) < \infty \quad (3.2)$$

where $J(\pi_i, b)$ is the jacobian of the map π_i. The first implication follows from the fact that $\pi_i(B_i) \subseteq D$ and the second follows from the change of variables formula and the fact that π_i is a diffeomorphism on the complement of a set of measure zero.

Theorem 4 implies that we can write $R(\xi) \circ \pi_i = R_i V_i$ where V_i is nowhere vanishing and bounded below and

$$R_i(\xi, Z) = \prod_{k=1}^n |Z_k|^{a_k\epsilon + b_k\delta + \sum_j c_{kj}\sigma_j}.$$

Here $a_k, b_k, c_{k,j} \in \mathbf{Z}$ may depend on b but are independent of ξ, and

$$|a_k| + |b_k| + |c_{k,j}| \leq H, \quad (3.3)$$

where H is a positive constant which depends only on the initial data defining $R(\xi, b, z)$ (the existence of H follows from Theorem 4). In particular, H does not depend on b.

Now we fix $I = (d_1, ..., d_n)$ such that $J(\pi_i, b) = v_i n_i = v_i(z)z^I$ as in Theorem 4, and let

$$F(Z) = \prod_{i=1}^{n} |Z_i|^{(a_i\epsilon + b_i\delta + \sum_j c_{ij}\sigma_j + d_i)}$$

Then we see that $\int_D R(\xi) < \infty$ implies $\int_{B_i} F < \infty$. Now $\int_{B_i} F < \infty$ if and only if

$$a_i\epsilon + b_i\delta + \sum_j c_{ij}\sigma_j > -2 - d_i \tag{3.4}$$

for all i. Let the fractional part $Frac(x) \in (0, 1]$ of a number x be defined by $x - Frac(x) \in \mathbf{Z}$. The key observation now is that, due to (3.3), the fractional parts of the numbers on the left hand side of (3.4) are always contained in the following set

$$\left\{ Frac(a_i\epsilon + b_i\delta + \sum_j c_{ij}\sigma_j) \ : \ |a_i| \ \le \ H, \ \ |b_i| \ \le \ H, |c_{ij}| \le H \right\},$$

which is a *finite set, independent of the parameter b in $R(\xi; b, z)$*. Let θ be the smallest element in this set. Then θ is in the range: $0 < \theta \le 1$. Set $\mu = \frac{\theta}{(n+2)H}$. Then μ is independent of the parameter b, and we see that (3.4) continues to hold if we replace ϵ, δ and the σ_j by ϵ', δ', σ_j' with

$$|\epsilon' - \epsilon|, |\delta' - \delta|, |\sigma_j' - \sigma_j| \le \mu.$$

Thus if $\xi' = (\epsilon', \delta', \sigma_j')$ and if we denote by $R_i(\xi', Z)$ the analogue of $R_i(\xi, Z)$ with ξ replaced by ξ', then

$$\int_{B_i} R_i(\xi', Z) \sum_{I, c_I \ne 0} |Z^I| \, dV(Z) < \infty \Longrightarrow \int_{\pi_i(B_i)} R(\xi')|J(\pi_i, b)| \, dV(Z) < \infty$$

$$\Longrightarrow \int_{\pi_i(B_i)} R(\xi', z) \, dV(z) < \infty \Longrightarrow \int_{D'} R(\xi') \, dV(z) < \infty$$

since $D' \subseteq \cup_{i=1}^{h} \pi_i(B_i)$. This completes the proof of the claim.

We can now complete the proof of Theorem 3. For $J \subseteq \{1, ..., n\}$ define

$$D_J = \{z = (z_1, ..., z_n) \in \mathbf{C}^n : |z_i| > 1 \ \text{ if and only if } \ i \in J\}$$

If $J = \emptyset$ then we write $D_J = D_0$. Note that D_0 is compact. Let ρ_J be the map $w \to z$ where $z_i = w_i^{-1}$ if $i \in J$ and $z_i = w_i$ if $i \notin J$. We observe that $\rho_J^{-1}(D_J) \subseteq D_0$ is a subset whose complement inside D_0 has measure zero.

Making the change of variables $z = \rho_J(w)$ we obtain

$$\begin{aligned}\int_{\mathbf{C}^n} R(\xi; b; z)\, dV &= \int_{\mathbf{C}^n} R(\xi; b; \rho_J(w)) \prod_{i \in J} |w_i|^{-4}\, dV \\ &= \int_{\mathbf{C}^n} R(J)(\Phi_J(\xi); b; w)\, dV\end{aligned} \tag{3.5}$$

where $R(J)(\xi; b; w) = R(P_i(J), Q_j(J); \Phi_J(\xi); b; w)$ for certain $P_i(J), Q_j(J) \in \mathbf{C}[b; w]$, and the map given by $\xi \mapsto \Phi_J(\xi) = (\epsilon, \delta, ...) \in \mathbf{R}^{n+2}$ is a certain invertible affine function (i.e., Φ_J is a linear automorphism of $\mathbf{R}^{n+2}$, composed with a translation). In particular, Φ_J is a homeomorphism.

Now assume that $I(\xi; b) < \infty$. Then, by (3.5),

$$\int_{\mathbf{C}^n} R(J)(\Phi_J(\xi); b; w)\, dV < \infty$$

The claim implies that there exists $\mu > 0$ such that for each J,

$$\int_{D_0} R(J)(\Phi_J(\xi'); b; w)\, dV < \infty \tag{3.6}$$

if $|\xi' - \xi| < \mu$. On the other hand, the change of variables formula implies

$$I(\xi'; b) = \sum_J \int_{D_0} R(J)(\Phi_J(\xi'); b; w)\, dV \tag{3.7}$$

which, when combined with (3.6), completes the proof of Theorem 3.

4. Proof of Theorem 1

We need a lemma relating uniform boundedness and continuity. The following lemma was proved in [PS1] (Lemma 5.4) when $\alpha = \beta$. The version we state here follows immediately by replacing α and β by $min(\alpha, \beta)$.

Lemma 4.1. *Let $D \subseteq \mathbf{C}^n$ be a compact domain and let $\mathcal{F} = \{(f, g)\}$ be a family of pairs of continuous functions on D with the property that $\{z : f(z) = 0\}$ and $\{z : g(z) = 0\}$ each have measure zero. Let $\alpha, \beta > 0$. Assume that*

$$\sup_{(f,g)\in\mathcal{F}} \int_D \frac{|f(z)|^{1-\alpha}}{|g(z)|^{1+\beta}} dV_1 \cdots dV_n < \infty$$

Let $(f_0(z), g_0(z)) \in \mathcal{F}$. Then for every $\tau > 0$ there exists $\rho > 0$ so that if $(f, g) \in \mathcal{F}$ and $\sup_D(|g - g_0| + |f - f_0|) < \rho$, then

$$\left| \int_D \frac{|f(z)|}{|g(z)|} dV_1 \cdots dV_n - \int_D \frac{|f_0(z)|}{|g_0(z)|} dV_1 \cdots dV_n \right| < \tau \ .$$

We shall also make use of the following elementary inequality: If A, B, α, β are positive numbers, then

$$\frac{A}{B} \leq \sum_{r=\pm1} \sum_{s=\pm1} \frac{A^{1+r\alpha}}{B^{1+r\beta}} \tag{4.1}$$

Now fix $\xi = (\epsilon, \delta, \sigma_1, ..., \sigma_n)$ as in the hypothesis of Theorem 1. We shall assume for the moment that $\xi \in \mathbf{Q}^{n+2}$. In view of (1.1),

$$R(\xi, b) = \frac{(\sum_i |P_i(b, z)|^{\epsilon})(\prod_t |z_t|^{\sigma_t^+})}{(\sum_j |Q_j(b, z)|^{\delta})(\prod_t |z_t|^{\sigma_t^-})} \tag{4.2}$$

where $\sigma_t^+ = \max(\sigma_t, 0)$ and $\sigma_t^- = |\min(\sigma_t, 0)|$. We shall also write $\xi = (\xi^+, \xi^-)$ where $\xi^+ = (\epsilon, \sigma_t^+)_{\{1 \leq t \leq n\}}$ and $\xi^- = (\delta, \sigma_t^-)_{\{1 \leq t \leq n\}}$. Thus ξ^+ contains the exponents which appear in the numerator of (4.2), and ξ^- the exponents in the denominator.

Now for α and β positive rational numbers we have

$$\begin{aligned} I(\xi, b) &= \int_{\mathbf{C}^n} R(\xi, b) \\ &= \sum_J \int_{D_0} R(\xi_J, b, z) \leq C \cdot \sum_J \sum_{r,s \in \{-1,1\}} \int_{\mathbf{C}^n} R(\xi_J(r, s), b, z) \end{aligned} \tag{4.3}$$

where $\xi_J(r, s) = (\xi_J(r, s)^+, \xi_J(r, s)^-)$, $\xi_J(r, s)^+ = (1 + r\alpha)\xi_J^+$, $\xi_J(r, s)^- = (1 + s\beta)\xi_J^-$, C is a positive constant, depending only on the initial data and $\xi_J = \Phi_J(\xi)$. The second equality follows from (3.5) and the inequality follows from (4.1).

Choosing α and β sufficiently small, and applying Theorem 3 to the function $R(\xi_J, b, z)$, we may assume that $I(\xi, b) < \infty$ if and only if the right side of (4.3) is finite.

Next we observe that $R(\xi_J(r, s), b, z)$ can be rewritten as an absolute rational power: Let $M \geq 2$ be an integer such that $M\xi \in \mathbf{Z}^{n+2}$. Write $\epsilon = e/M, \delta = d/M, \sigma_t = s_t/M$ and let $\tilde{P} = P^e \prod_{s_t>0} z^{s_t}$, $\tilde{Q} = Q^d \prod_{s_t<0} z^{|s_t|}$. Then $R(\xi_J(r, s), b, z) = R(\epsilon(r, s), \delta(r, s), \tilde{P}, \tilde{Q})$ where $(\epsilon(r, s), \delta(r, s)) = (\epsilon_J(r, s), \delta_J(r, s)) = ((1 + r\alpha)/M, (1 + r\beta)/M)$.

The pair $(\epsilon(r, s), \delta(r, s))$ is non-degenerate (see the statement of Theorem 2 for the definition of non-degenerate) if and only if $\mu((1 + r\alpha)/M) - \nu((1 + s\beta)/M) = 2$ for some $\mu, \nu \in \mathbf{Z}$ such that $0 \leq \mu \leq deg(\tilde{P})$ and $0 \leq \nu \leq deg(\tilde{Q})$. Thus the pair is non-degenerate if (α, β) lies in the complement of a finite union of lines in $\mathbf{Q}^2$. Choosing such a pair we may assume that $(\epsilon_J(r, s), \delta_J(r, s))$ is non-degenerate for all r, s and all J. Thus we may apply Theorem 2 to the dV_n integrals which appear on the right side of (4.3):

$$\int_{\mathbf{C}} R(\xi_J(r, s), b, z) dV_n \sim T_{\lambda(b; z_1, ..., z_{n-1})}(b; z_1, ..., z_{n-1}) \tag{4.4}$$

Letting $\tilde{\lambda}(b) = \inf_{(z_1,...,z_{n-1})\in\mathbf{C}^{n-1}} \lambda(b; z_1, ..., z_{n-1})$, we obtain

$$\begin{aligned}&\int_{\mathbf{C}} R(\xi_J(r, s); b; z_1, ..., z_n)\, dV_n \sim T_{\tilde{\lambda}(b)}(b; z_1, ..., z_{n-1})\\ &\text{for almost all } (z_1, ..., z_{n-1}) \in \mathbf{C}^{n-1}.\end{aligned} \tag{4.5}$$

Fox λ fixed, let $W_\lambda = \{b : \tilde{\lambda}(b) = \lambda\}$. Then W_λ is constructible: To see this, we define $V_\lambda = \{(b, z_1, ..., z_{n-1}) : \lambda(b, z_1, ..., z_{n-1}) \geq \lambda\}$ (where $\lambda(b, z_1, ..., z_{n-1})$ is as in (4.4)). Thus $\mathcal{B} \times \mathbf{C}^{n-1} = V_0 \supset V_1 \supset \cdots \supset V_{\mathcal{N}} = \emptyset$ is a filtration by algebraic subvarieties. Let $\pi : \mathcal{B} \times \mathbf{C}^{n-1} \to \mathcal{B}$ be the projection onto the first factor, and let $S_j = \pi(V_j \backslash V_{j+1})$. Then S_j is constructible since it is the image of a constructible set by an algebraic morphism. Moreover, since $W_\lambda = \{b \in S_\lambda : b \notin S_j \text{ for } j < \lambda\}$, we see that W_λ is also constructible.

Now for all $b \in W_\lambda$, equation (4.5) implies:

$$\begin{aligned}&\int_{\mathbf{C}^n} R(\xi_J(r, s); b; z_1, ..., z_n)\, dV_n dV_{n-1} \cdots dV_1\\ &\sim \int_{\mathbf{C}^{n-1}} T_\lambda(b; z_1, ..., z_{n-1}) dV_{n-1} \cdots dV_1\end{aligned} \tag{4.6}$$

But the right side of (4.6) is an integral of an ARP in $n - 1$ variables, and hence we may use induction to decompose W_λ into constructible pieces in such a way that the integral on the right side is either infinite or finite and continuous on each piece.

Thus we have proved the following: For ξ fixed as above, the set $\mathcal{B}_i = \{b \in \mathcal{B} : I(\xi, b) = \infty\}$ is constructible. Moreover, the set $\mathcal{B}_f = \{b \in \mathcal{B} : I(\xi, b) < \infty\}$ can be decomposed into constructible pieces B_μ in such a way that for each μ we have, by (4.3),

$$\int_{D_0} R(\xi_J(1, -1), b, z) \;\leq\; F_\lambda(b)$$

for some finite continuous function F_λ. Lemma 4.1 now implies that $I(\xi, b)$ is continuous for $b \in B_\mu$, and this completes the proof of Theorem 1.

5. Controlled uniformization

In this section we use the algorithm of [BM88a][BM88b] to give the proof of Theorem 4. We shall need a few lemmas from point set topology:

Lemma 5.1. *Let M be a manifold and $\{U_\alpha : \alpha \in A\}$ a basis of open sets of M. Then there exists $B \subseteq A$ such that $\{U_\alpha \hookrightarrow M : \alpha \in B\}$ is a locally finite cover of M.*

Lemma 5.2. *Let $\{\pi_i : M_i \to M\}$ be a locally finite cover of M and, for each i, suppose we are given a locally finite cover $\{\pi_{ij} : M_{ij} \to M_i : j \in J(i)\}$. Then* *there is a finite subset* $F(i) \subseteq J(i)$ *such that the family* $\{\pi_i \circ \pi_{ij} : M_{ij} \to M : j \in F(i)\}$ *is a locally finite cover of* M.

Lemmas 5.1 and 5.2 allow us to assume, in the proof of Theorem 4, that the original M is a coordinate neighborhood, that is, that M is a polydisk in $\mathbf{C}^n$.

Theorem 4 says that if f is a polynomial, then the degrees of the monomials which appear in its uniformization are bounded in terms of the degree of f. To prove this theorem, we use the [BM88a][BM88b] algorithm which is an inductive procedure where the starting point of the uniformization of f, which is a function of n variables, is the uniformization of the coefficients of its Weierstrass polynomial, which are functions of $n-1$ variables. Of course these coefficients are not themselves polynomials, which prevents us from using straightforward induction. On the other hand, the coefficients are *nearly* polynomials in the sense that they are “algebraic" . Thus we are led to a slightly more general formulation of Theorem 4 in which the function f is allowed to be an arbitrary algebraic function. First we recall the definition of a (single-valued) algebraic function:

Let $M \subseteq \mathbf{C}^n$ be an open subset. Then we say that $f : M \to \mathbf{C}$ is an (single-valued) algebraic function of degree at most k if f is holomorphic and if f satisfies a non-zero polynomial equation $F(t) = 0$, where $F(t) \in \mathbf{C}[x_1, ..., x_n, t]$ is of degree at most k, that is:

$$\sum_{\mu=0}^{k} A_\mu(x_1, ..., x_n) f(x)^{k-\mu} \; = \; 0 \tag{5.1}$$

where $A_\mu \in \mathbf{C}[x_1, ..., x_n]$ has degree at most μ.

Algebraic functions behave well with respect to Weierstrass preparation, i.e., the coefficients of the Weierstrass polynomial of an algebraic function f are themselves algebraic, and their degrees are bounded in terms of the degree of f. More generally,

Lemma 5.3. *Let* $f : M \to \mathbf{C}$ *be holomorphic with* $M \subseteq \mathbf{C}$ *open and connected.*

1. *Assume that* f *is algebraic and that* $f = gh$ *where* g *and* h *are holomorphic functions on* M. *Assume as well that there is a point* $p \in M$ *and a polydisk* $D = D_1 \times \cdots \times D_n$ *centered at the origin such that* $p + D \subseteq M$ *and such that*
   - a) *The function* $g|_D(x + p)$ *is a monic polynomial in* x_n *whose coefficients are holomorphic functions in the remaining variables, that is,* $g^*(x) = g|_D(x + p) = \sum_{i=0}^{d} B_i(\tilde{x}) x_n^{d-i}$ *where* $\tilde{x} = (x_1, ..., x_{n-1})$, $B_i(\tilde{x})$ *is holomorphic on* $D_1 \times \cdots \times D_{n-1}$ *and* $B_0 = 1$.
   - b) *For each fixed* $\tilde{x} \in D_1 \times \cdots D_{n-1}$, *all the roots of the equation* $g^*(\tilde{x}, x_n) = 0$ *are in the polydisk* D_n.

   Then g, $B_i(\tilde{x})$ *and* h *are all algebraic. Moreover, their degrees are bounded in terms of the degree of* f.

2. *Assume that f satisfies a polynomial $\sum_{\mu=0}^{k} B_\mu(x_1, ..., x_n) f(x)^{k-\mu} = 0$ where the B_μ are algebraic functions on M. Then f is algebraic and its degree is bounded in terms of the degrees of the B_μ.*
3. *Assume that f is algebraic and the $\pi : B \to M$ is algebraic (i.c. its components are algebraic) with B a polydisk. Then $f \circ \pi$ is algebraic and $deg(f \circ \pi)$ is bounded in terms of $deg(f)$ and $deg(\pi)$.*

We shall need the following Lemma:

Lemma. Let $m, n > 0$. Then there exist polynomials $P_i, Q_j \in \mathbf{Z}[A_1, ..., A_m, B_1, ..., B_n]$, $1 \leq i, j \leq r = mn$, with the following property: If $R \subseteq S$ are commutative rings and $\alpha, \beta \in S$ satisfy polynomials

$$\alpha^m + a_1\alpha^{m-1} + \cdots a_m = 0 \text{ and } \beta^n + b_1\beta^{n-1} + \cdots \beta_n = 0$$

then $\sigma = \alpha + \beta$ and $\pi = \alpha\beta$ satisfy the polynomials:

$$\sigma^r + \sum_{i=1}^{r} P_i(a_1, ..., a_m, b_1, ..., b_n)\sigma^{r-i} = 0$$

$$\text{and} \quad \pi^r + \sum_{j=1}^{r} Q_j(a_1, ..., a_m, b_1, ..., b_n)\pi^{r-j} = 0$$

We recall the proof, which is well known (see [L] for example):

Proof. Let $M \subseteq S$ be the R-module generated by the elements

$$\{\alpha^i\beta^j : 0 \leq i \leq m-1, 0 \leq j \leq n-1\} = \{\mu_1, ..., \mu_r\}$$

where $r = mn$. The fact that α and β satisfy monic polynomials implies $\alpha M \subseteq M$ and $\beta M \subseteq M$. If $t = \alpha + \beta$ or if $t = \alpha\beta$, then $tM \subseteq M$. Thus for every k, $1 \leq k \leq r$, we have

$$t\mu_k = \sum_{l=1}^{r} c_{kl}\mu_l$$

for some $c_{kl} \in R$. The c_{kl} are easily seen to be polynomials in the a_i and b_j which depend only on m and n. Let C be the matrix $C = c_{kl}$. Then $(C - tI)\mu = 0$, where μ is the column vector whose k^{th} entry is μ_k. Multiplying both sides of by the adjoint of $C - tI$, we obtain: $det(C - tI)\mu_k = 0$ for every k. Since one of the $\mu_k = 1$, this proves $det(C - tI) = 0$. Finally we observe that the characterstic polynomial of C is monic in t with coefficients which are polynomials in the a_i and b_j which depend only on m and n. This proves the lemma.

Proof of Lemma 5.3. We start with the proof of part 1: We may assume that $p = 0$ since $f(x)$ is algebraic if and only if $f(x + p)$ is algebraic. Moreover, if $f(x)$ is algebraic, then $\deg(f(x)) = \deg(f(x + p))$. Furthermore, since M

is connected, we may assume that $M = D$ (if a holomorphic function on M satisfies a polynomial identity on some non-empty open subset of M then, by analytic continuation, the identity holds on all of M).

Let $\alpha_1(\tilde{x}), ..., \alpha_d(\tilde{x})$ be the roots of $g(\tilde{x}, x_n) = 0$ (counted with multiplicity). By choosing a smaller polydisk $D'_1 \times \cdots D'_{n-1} \subseteq D_1 \times \cdots D_{n-1}$ (possibly changing the center) we may assume that the α_i are all single valued holomorphic functions of $\tilde{x} \in D'_1 \times \cdots D'_{n-1}$.

Equation (5.1) implies that f divides A_k, which we may assume to be non-zero (otherwise divide both sides of (5.1) by the appropriate power of f in order to obtain a polynomial whose A_k term is non-zero). Since g divides f we conclude that the d roots of $g(\tilde{x}, x_n) = 0$, $\alpha_1(\tilde{x}), ..., \alpha_d(\tilde{x})$ (counted with multiplicity), form a subset of the roots of

$$A_k(\tilde{x}, x_n) \;=\; \sum_{r=0}^{k} c_r(\tilde{x}) x_n^{d-r} \;=\; 0$$

After shrinking the polydisk further if necessary, we may assume that $c_0(\tilde{x})$ is nowhere vanishing.

Now let $A'(\tilde{x}, y_n) = c_0^k A(\tilde{x}, y_n/c_0)$. Then $A'(\tilde{x}, y_n) = A'(y_n)$ is monic in y_n of degree k, and the $\alpha'_i = c_0\alpha_i$ all satisfy $A'(y_n) = 0$. The lemma now implies that the elementary symmetric polynomials in the α'_i satisfy monic polynomials of bounded degrees which implies that the elementary symmetric polynomials in the α_i satisfy non-zero (but not necessarily monic) polynomials of bounded degrees. This proves that the $B_i(\tilde{x})$ are algebraic of bounded degree. A similar argument shows that g and h are also algebraic of bounded degree, and part 1 of Lemma 5.3 is proved.

Proof of Part 2. After multiplying both sides of $\sum_{\mu=0}^{k} B_\mu(x) f(x)^{k-\mu} = 0$ by the appropropriate polynomial, we may assume that the B_μ are integral, that is, they satisfy monic polynomials with coefficients in the ring $R = \mathbf{C}[x_1, ..., x_n]$. Now let $F(Y) = \sum_\mu B_\mu Y^\mu$, let $K = \mathbf{C}(x_1, ..., x_n)$, let $\bar{K}$ be the algebraic closure of K and let G be the Galois group of $\bar{K}$ over K. Then G naturally acts on the set of polynomials with coefficients in $\bar{K}$. Let $F_1, ..., F_N$ be the orbit of F under the action of G. Let $\Phi(Y) = \prod_i F_i(Y)$. Then $\Phi(Y) \in K[Y]$ and its coefficients are integral over R. Thus $\Phi(Y) \in R[Y]$. Moreover, by part 1 of Lemma 5.3, the coefficients of $\Phi(Y)$ have degrees which are bounded in terms of the degrees of the B_μ and $\Phi(f(x)) = 0$. This proves part 2 of Lemma 5.3.

To prove part 3, assume that f satisfies equation (5.1). Then $f \circ \pi$ satisfies the equation $\sum_\mu A_\mu(\pi(x))(f \circ \pi)(x)^\mu = 0$. This is a polynomial with algebraic coefficients whose degrees are bounded in terms of the degree of f and the degree of π. Thus, we may apply part 2 to conclude that $f \circ \pi$ is algebraic and that $deg(f)$ is bounded in terms of $deg(f)$ and $deg(\pi)$. This concludes the proof of Lemma 5.3.

Now the idea for the proof of Theorem 4 is as follows: We write f as a unit times a Weierstrass polynomial. Lemma 5.3 allows us to apply induction to the coefficients of the Weierstrass polynomial which reduces us, via Lemmas 5.1 and 5.2, to the case where f is a polynomial with monomial coefficients. We then perform a sequence of local blow-ups with smooth centers until we arrive at a factorization of f as a Weierstrass polynomial of lower degree, multiplied by a monomial. Continuing in this fashion, until the degree of the Weierstrass polynomial reduces to zero, we eventually see that f can be written in local coordinates as a monomial times a nowhere vanishing factor.

In order to give a precise description of the induction procedure described above, we require some additional notation:

If g is a holomorphic function in an open neighborhood of a point $b \in \mathbf{C}^n$, then $mult_b(g)$ is the degree of the lowest order terms appearing in the Taylor expansion of g at b. Thus $mult_b(g) = 0$ if and only if $g(b) \neq 0$.

Let $\mathbf{N} = \{0, 1, 2, ...\}$ be the set of natural numbers. If $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbf{N}^k$, then $|\alpha| = \alpha_1 + \cdots \alpha_k$. If $\alpha, \beta \in \mathbf{N}^k$ then we write

$$\alpha \leq \beta \text{ if and only if } \alpha_i \leq \beta_i, \ \ 1 \leq i \leq k.$$

Let $f : M \to C$ be an algebraic function, where $M \subseteq \mathbf{C}$ is open and connected. Let $\pi : B_p \to M$ be an algebraic map (i.e., all the components of π are algebraic), where B_p is an open polydisk centered at p and let $n, D, R, N \in \mathbf{N}$ with $n > 0$. We say that π is of type $\mathcal{T} = (n, D, R, N)$ if $f \circ \pi$ and $Jac(\pi)$ (the jacobian determinant of π) can be factored as follows:

$$\begin{aligned} (f \circ \pi)(x + p) &= u(x) x^{\xi} \lambda_1(x)^{\mu_1} \cdots \lambda_r(x)^{\mu_r} g(x), \\ Jac(\pi)(x + p) &= v(x) x^{\theta} \lambda_1(x)^{\nu_1} \cdots \lambda_r(x)^{\nu_r} \end{aligned} \tag{5.2}$$

where

(1) $\deg(f \circ \pi) \leq N$ and $\deg(\pi) \leq N$ (that is, each component of π has degree at most N).
(2) u and v are algebraic on $B = \{x \in \mathbf{C}^n : x + p \in B_p\}$, and are uniformly bounded below, that is, there exists $c > 0$ such that $|u(x)| \geq c$ and $|v(x)| \geq c$ for all $x \in B$.
(3) $x^{\xi} = \prod_{h=1}^{n} x_h^{\xi_h}$ where $\xi = (\xi_1, ..., \xi_n) \in \mathbf{N}^n$ (and similarly for x^{θ}).
(4) r is an integer such that $0 \leq r \leq R$, $\mu = (\mu_1, .., \mu_n) \in \mathbf{N}^r$, $\nu = (\nu_1, .., \nu_n) \in \mathbf{N}^r$, $\lambda_i(x) = x_n + a_i(\tilde{x}),\ i = 1, ..., r$, with $\tilde{x} = (x_1, ..., x_{n-1})$ and the $a_i(\tilde{x})$ distinct non-zero algebraic functions.
(5) g is algebraic and $\sup_{b \in B} \mathrm{mult}_b(g(x)) \leq D$. Let $s \in \mathbf{N}$. Then π is of type $\mathcal{W} = (n, D, R, N, s)$ if it is of type (n, D, R, N) and if, in addition to conditions (1),(2),(3),(4) and (5), the following conditions hold:

(6) For some integer $d \leq D$,

$$g(x) = w(x)(x_n^d + \sum_{j=2}^{d} c_j(\tilde{x})x_n^{d-j}) \tag{5.3}$$

with w algebraic and uniformly bounded below, $c_j(\tilde{x})$ algebraic, and for every j such that $c_j \neq 0$ we have $c_j(\tilde{x})^{d!/j} = v_j(\tilde{x})\tilde{x}^{\beta_j}$ for some $\beta_j \in \mathbf{N}^{n-1}$ with $v_j(\tilde{x})$ algebraic and uniformly bounded below.

(7) $a_i(\tilde{x})^{d!} = u_i(\tilde{x})\tilde{x}^{\alpha_i}$ $i = 1, ..., r$ for some $\alpha_i \in \mathbf{N}^{n-1}$ where u_i is algebraic and uniformly bounded below.

(8) $\{\alpha_i, \beta_j\}$ is totally ordered, with $\alpha_i \leq \alpha_{i+1}$ for all i, $1 \leq i \leq r-1$.

(9) $|\inf\{\alpha_i, \beta_j\}| \leq s$

Remark 5.1. If f is of type $\mathcal{W} = (n, D, R, N, s)$ and if $a_i(0) = c_j(0) = 0$ for all i, j, then Lemma 5.3 implies that the values of $|\xi|$, $|\theta|$, $|\alpha_i|$ and $|\beta_i|$ are all bounded in terms of N.

Remark 5.2. The fact that $c_1 = 0$ in (5.3) implies that $mult_b(g) \leq d$ for all b with equality if and only if $b_n = 0$ and $mult_b(c_j(\tilde{x})) \geq j$ for all j.

Remark 5.3. If π is of type $\mathcal{T} = (n, 0, 0, N)$ or of type $\mathcal{W} = (n, 0, 0, N, s)$ then $f \circ \pi$ and $Jac(\pi)$ are monomials (up to a nowhere vanishing holomorphic factor).

If $\mathcal{T} = (n, D, R, N)$ then we write $|\mathcal{T}| = n + D + R + N$. Similarly, if $\mathcal{W} = (n, D, R, N, s)$ then $|\mathcal{W}| = n + D + R + N + s$.

Lemma 5.4. *Assume $f : M \to \mathbf{C}$ is an algebraic function on $M \subseteq \mathbf{C}^n$, a connected open set, and let $\pi : B \to M$ be an algebraic map, where $B \subseteq \mathbf{C}^n$ is a centered open polydisk.*

1. *If π is of type $\mathcal{T} = (n, D, R, N)$ and if $q \in B$ is such that $q_n \neq 0$ then for every sufficiently small neighborhood U_q containing q, there is a polydisk B_q centered at q and an algebraic map $\pi_q : B_q \to U_q$ such that $\pi_q(q) = q$, π_q maps B_q biholomorhically onto its image, and $\pi \circ \pi_q|_{B_q}$ has type $\mathcal{T} = (n, D', R', N')$ satisfying the following: If $D \neq 0$ then $D' < D$. If $D = 0$ and $R \neq 0$ then $D' = 0$ and $R' < R$. Moreover, $|\mathcal{T}'|$ is bounded above by a constant which depends only on $|\mathcal{T}|$.*
2. *If π is of type $\mathcal{T} = (n, D, R, N)$ then there exists a locally finite covering of B of the form $\{\pi_j : B_j \to B\}$, with the $B_j \subseteq \mathbf{C}^n$ centered open polydisks, and the π_j algebraic such that for each integer j, $\pi \circ \pi_j$ is of type $\mathcal{W} = (n, D', R', N', s')$ where $|\mathcal{W}|$ is bounded in terms of $|\mathcal{T}|$. If $D = 0$ then $R' = R$.*
3. *If π is of type $\mathcal{W} = (n, D, R, N, s)$ with $s \geq D!$ then there exists $\{\pi_j : B_j \to B\}$, a locally finite covering, with the $B_j \subseteq \mathbf{C}^n$ centered open polydisks, and π_j algebraic such that for every j, $\pi \circ \pi_j$ is of type $\mathcal{W}' = (n, D', R', N', s')$*

where $(D', R', s') < (D, R, s)$ *with respect to the lexicographic ordering, and* $|\mathcal{W}'|$ *is bounded in terms of* $|\mathcal{W}|$.

Before proving the lemmas, we first show how they can be used to prove Theorem 4:

Proof of Theorem 4. Let f be an algebraic function on an open set $M \subseteq \mathbf{C}^n$ of degree d_0. As noted in the proof of Lemma 5.3, we observe that f divides $A_0(x)$ and thus the multiplicity of f at any point is bounded above by the degree of f. Thus, using Lemma 5.1 and Lemma 5.2, we can find a locally finite cover of M with the property that each π in the covering has type $\mathcal{T} = (n, d_0, 0, d_0)$, and thus $\mathcal{T}$ depends only on d_0 (and n, which is fixed).

Now by part (2) of Lemma 5.4, we may assume that each π has type $\mathcal{W} = (n, D, R, N, s)$ where $|\mathcal{W}|$ is bounded in terms of $|\mathcal{T}|$ (and hence in terms of d_0).

Thus we are led to make the following assertion:

Claim. Suppose $\pi : B \to M$ is an algebraic map of type $\mathcal{W} = (n, D, R, N, s)$. Then there exists a locally finite algebraic covering $\{\pi_j : B_j \to B\}$ such that for every j we have: $\pi \circ \pi_j$ is of type $(n, 0, 0, N')$ where N' depends only on $\mathcal{W}$.

Note that in view of Lemma 5.3, the claim immediately implies parts (1), (2) and (3) of Theorem 4. Part (4) (which is not needed in this paper) will follow from the construction of π_i.

We prove the claim arguing by induction on the triple (n, D, R) (with respect to the lexicographic ordering). Since the assertion is clear in the case $n = 1$, we shall assume $n > 1$. We may also assume in equation (5.3) that $d = D$, and that in equation (5.2) we have $r = R$.

By part (3) of Lemma 5.4, we may assume that π has type $\mathcal{W} = (n, D, R, N, s)$ with $s < D!$. This means that $f \circ \pi$ has a factorization (5.2) satisfying conditions (1) through (9) with either $\alpha_i = 0$ for some i or $mult_0(c_j(\tilde{x})) < j$ for some j. The fact that B is centered implies that $p = 0$.

Case $\alpha_i = 0$

Without loss of generality we may assume that $i = 1$. To complete the inductive step, we must prove the following claim: B has a locally finite cover $\pi_j : B_j \to B$ by algebraic maps such that for each j, $\pi \circ \pi_j$ has type $\mathcal{W}' = (n, D', R', N', s')$ where $\mathcal{W}'$ depends only on $\mathcal{W}$ and where $D' < D$ or $D = D'$ and $R' < R$.

Lemmas 5.1 and 5.2 imply that we can replace B by the elements in some basis of the open sets of B. Part 1 of Lemma 5.4 shows that the claim is true for some basis of the open sets of $\{q \in B : q_n \neq 0\}$. Thus is suffices to show that for every point $q \in B$ with $q_n = 0$, the claim is true for the function $f|_{B_q}$ for all sufficiently small polydisks centered at q. But for such a q the factor $\lambda_1(x)$ does not vanish at $x = q$, which means that on every sufficiently small polydisk centered at q, in the factorization (5.2) the factor $u(x)$ may be replaced by $u(x)\lambda_1(x)^{\mu_1}$. Thus on that polydisk, π has type $\mathcal{W}' = (n, D, R-1, N, s')$ where $\mathcal{W}'$ depends only on $\mathcal{W}$ (in this case π_j is the identity).

Case: $mult_0(c_j(\tilde{x})) < j$ for some j
Remark 5.2 implies $mult_b(g(x)) < D$ for all $b \in B$. Thus π has type $\mathcal{T} = (n, D-1, R, N)$. Using part 2 of Lemma 5.4 we may assume that π has type $\mathcal{W}' = (n, D-1, R', N', s')$ where $\mathcal{W}'$ depends only on $\mathcal{W}$.

This completes the inductive step and shows that Theorem 4 follows from Lemmas 5.1, 5.2, 5.3 and 5.4. Q.E.D.

Proof of Lemma 5.4. This lemma is essentially proved in [BM89]. We shall reproduce their proof, making the necessary modifications, and keeping track of the exponents in order to obtain the desired bounds.

We start with part 1): Let $q \in B$ such that $q_n \neq 0$. If $D > 0$, then, by remark 5.2, we conclude that for B_q sufficiently small, $\pi|_{B_q}$ has type $\mathcal{T} = (n, D-1, R, N)$. Thus we may assume that $D = 0$. Note that $D = 0$ implies that $g = 1$.

If $\lambda_1(x) = x_n + a_1(\tilde{x})$ does not vanish at $x = q$, then, after possibly shrinking the radius of B_q, we may further assume that $|\lambda_1(x)|$ is bounded below by a positive constant, and thus, replacing u by $u\lambda_1^{\mu_1}$, we see that f has type $\mathcal{T} = (n, 0, R-1, N)$.

If $\lambda_1(x) = x_n + a_1(\tilde{x})$ does vanish at $x = q$, then we can replace x_n by λ_1 thus reducing R to $R-1$. To be precise, define $\pi_q(y_1, ..., y_n) = (x_1, ..., x_n)$ by the formula: $x_n = y_n - a_1(\tilde{y}) - q_n$ and $x_i = y_i$ for $i < n$. Then $\pi_q : B_q \to B$ for sufficiently small polydisks B_q, centered at q and $\pi_q(q) = q$. We easily see that π_q is an algebraic map and is a homeomorphism of B_q onto its image. Moreover we have factorizations:

$$f((\pi \circ \pi_q)(y+q)) = u'(y) y^{\xi'} \lambda_2'(y)^{\mu_2} \cdots \lambda_r'(y)^{\mu_r}$$

$$J(\pi \circ \pi_q)(y+q) = v'(y) y^{\theta'} \lambda_2'(y)^{\nu_2} \cdots \lambda_r'(y)^{\nu_r}$$

where:

$$u'(y) = u(\pi_q(y+q)) \prod_{\{1 \le h \le n-1 : q_h \neq 0\}} (y_h + q_h)^{\xi_h}$$

$$y^{\xi'} = \left(\prod_{\{1 \le h \le n-1 : q_h = 0\}} (y_h)^{\xi_h} \right) y_n^{\mu_1}$$

$$\lambda_j'(y) = (y_n - a_1(\tilde{y} + \tilde{q}) + a_j(\tilde{y} + \tilde{q})) \text{ for } 2 \le j \le r$$

with similar formulas for $v'(y)$ and $y^{\theta'}$. Lemma 5.3 now implies that $(\pi \circ \pi_q)$ has type $\mathcal{T}' = (n, 0, R-1, N')$ where $\mathcal{T}'$ depends only on $\mathcal{T}$. This completes the proof of part (1).

Now we prove part (2): Assume f is of type $\mathcal{T} = (n, D, R, N)$. Let $q \in B$ and let $x = y + q$. Then (5.2) becomes:

$$(f \circ \pi)(y + q) = u(y + q) \prod_{i=1}^{n} (y_i + q_i)^{\xi_i} \prod_{i=1}^{r} (y_n + a_i'(\tilde{y}))^{\mu_i} g'(y)$$

$$Jac(\pi)(y + q) \;=\; v(y + q) \prod_{i=1}^{n} (y_i + q_i)^{\theta_i} \prod_{i=1}^{r} (y_n + a_i'(\tilde{y}))^{\nu_i} \tag{5.4}$$

for $y \in B'$, a sufficiently small ball centered at 0. Here we set $a_i'(\tilde{y}) = q_n + a_i(\tilde{y} + \tilde{q})$ and $g'(y) = g(y + q)$.

Let $\lambda_i'(y) = y_n + a_i'(\tilde{y})$ and order the λ_i' in such a way that $\lambda_i'(0) \neq 0$ if and only if $i > r'$. Let $u'(y) = u(y + q) \prod_{q_i \neq 0} (y_i + q_i)^{\xi_i} \prod_{i > r'} \lambda_i'(y)$ and similarly for $v'(y)$. Then

$$(f \circ \pi)(y + q) = u'(y) \tilde{y}^{\xi'} \prod_{i=1}^{r'} \lambda_i'(y)^{\mu_i} g'(y)$$

where $\xi_i' = \xi_i$ if $q_i = 0$ and $\xi_i' = 0$ otherwise.

Let $\rho : \mathbf{C}^n \to \mathbf{C}^n$ be a generic rotation. Then $\rho : B'' \to B'$ for B'' a sufficiently small centered polydisk. The Weierstrass Preparation Theorem implies

$$g'(\rho(z)) \;=\; w_0(z) \left(z_n^d + \sum_{j=1}^{d} b_j(\tilde{z}) z_n^{d-j} \right), \quad \text{and } \lambda_i'(\rho(z)) \;=\; w_i(z)(z_n + a_i''(z)) \tag{5.5}$$

for $z \in B''$ (possibly after shrinking B'' further). Here the $w_i(z)$ are nowhere vanishing holomorphic functions, $b_j(\tilde{z})$ is holomorphic and satisfies $\text{mult}_0 b_j \geq j$, $a_i''(\tilde{z})$ is holomorphic and $a_i''(0) = 0$.

Lemma 5.3 implies the b_j and a_i'' are algebraic with degrees which are bounded in terms of $\mathcal{T}$. Equation (5.4) now becomes:

$$(f \circ \pi)(\rho(z) + q) \;=\; u''(z) \prod_{i=1}^{r''} (z_n + a_i''(\tilde{z}))^{\mu_i''} p(z)$$

where $r'' = r' + n$ and $p(z)$ is the Weierstrass polynomial which appears in (5.5) and u'' is nowhere vanishing and bounded away from zero. Here $\mu_i'' = \mu_i$ if $i \leq r'$ and $\mu_i'' = \xi_{i-r'}'$ if $r' < i \leq r''$. The function $(Jac(\pi))(\rho(z) + q)$ has a similar expression - u'' is replaced with v'', μ_i'' is replaced with ν_i'' and the factor $p(z)$ is omitted.

Let $\tau(w) = (w_1, ..., w_{n-1}, w_n - b_1(\tilde{w})/d)$. Then $\tau : B''' \to B''$ where B''' is a sufficiently small ball centered at 0. Let $\pi_q(w) = \rho(\tau(w)) + q$. Then

$$(f \circ (\pi \circ \pi_q))(w) \;=\; u'''(w) \prod_{i=1}^{r'} (w_n + a_i'''(\tilde{w}))^{\mu_i''} p'(w)$$

$$J(\pi \circ \pi_q))(w) \;=\; v'''(w) \prod_{i=1}^{r'} (w_n + a_i'''(\tilde{w}))^{\nu_i''}$$

where $p'(w) = w_n^d + \sum_{i=2}^{d} c_j(\tilde{w}) w^{d-j}$ and $u'''(w)$, $v'''(w)$ are bounded away from zero.

Lemma 5.3 implies that the c_j and the a_i''' are algebraic with degrees bounded in terms of $\mathcal{T}$. Lemma 5.1 implies that a subset of the collection of $\pi_q : B''' \to B$ obtained in this fashion forms a locally finite cover of B.

Now let $h(\tilde{x})$ denote the product of all the non-zero functions in the following list, and all of their non-zero differences : $(a_i''')^{d!}$, $c_j^{d!/j}$. The degree of h is bounded in terms of $\mathcal{T}$.

Now $h(\tilde{x})$ is an algebraic function of $n - 1$ variables, so, by induction, the conclusion of the theorem applies: After passing to a locally finite cover, the a_i''' and the c_j are monomials (up to multiplication by a unit) whose degrees are bounded in terms of $\mathcal{T}$. To show that f is of type $\mathcal{W}$, it only remains to verify that condition (8) holds. But this is guaranteed by the following lemma of Bierstone and Milman:

Lemma 5.5. ([BM89], Lemma 3.4). *Let* $z = (z_1, ..., z_n)$. *Let* $\alpha, \beta \in \mathbf{N}^n$ *and let* $a(z), b(z), c(z)$ *be formal power series with non-zero constant terms. If*

$$a(z)z^{\alpha} - b(z)z^{\beta} = c(z)z^{\gamma}$$

then either $\alpha \leq \beta$ *or* $\beta \leq \alpha$.

The proof of part 2 of Lemma 5.4 is complete.

Next we prove part 3 of Lemma 5.4. Assume π is of type $\mathcal{W} = (n, D, R, N, s)$ with $s > D!$. For each $q \in B$ let $B_q(r) = \prod_{i=1}^{n} \{z \in \mathbf{C}^n; |z_i - q_i| < r\} \subseteq B$. Parts 1 and 2 of Lemma 5.4, imply that for r sufficiently small:

1. If $q_n \neq 0$ and $D > 0$ then $B_q(r)$ has a locally finite covering $\pi_i : B_i \to B_q(r)$ such that $\pi \circ \pi_i$ is of type $\mathcal{W}' = (n, D - 1, R', N', s')$ with $\mathcal{W}'$ depending only on $\mathcal{W}$.
2. If $q_n \neq 0$ and $D = 0$ and $R > 0$ then $B_q(r)$ has a locally finite covering $\pi_i : B_i \to B_q(r)$ such that $\pi \circ \pi_i$ is of type $\mathcal{W}' = (n, 0, R - 1, N', s')$ where $\mathcal{W}'$ depends only on $\mathcal{W}$.
3. If $q_n = 0$ then $\pi|_{B_q(r)}$ has type $\mathcal{W} = (n, D(q), R(q), N, s)$, where $D(q) = mult_q(g)$ and $R(q) = |\{i : \lambda_i(q) = 0\}|$.

Lemma 5.1 now implies that there is a subset $A \subseteq B$ and real numbers $r(q) > 0$ for $q \in A$ such that $\{B_q(r(q)/2) : q \in A\}$ forms a locally finite cover of B and such that $B_q(r(q))$ satisfies 1 or 2 or 3 for every $q \in A$. We may also assume that the $r(q)$ are chosen in such a way that $\lambda_i(q) \neq 0$ implies $\lambda_i(x) \neq 0$ for all $x \in B_q(r(q))$.

Observe that $\{B_q(r(q)) : q \in A\}$ must also form a locally finite cover of B. We shall write $B_q = B_q(r(q))$ and $B'_q = B_q(r(q)/2)$

If $q \in A$ such that $q_n \neq 0$, then, using (1) and (2) we see that $\pi|_{B'_q}$ satisfies the conclusion of part 3 of Lemma 5.4.

Now let $q \in A$ be such that $q_n = 0$. If $mult_q(g) < D$ or if $\lambda_i(q) \neq 0$ for some i, then part 2 of Lemma 5.4 implies that $f|_{B_q(r)}$ has type $\mathcal{W} = (n, D', R', N', s')$ with $(D', R') < (D, R)$ and hence it satisfies the conclusion of part 3 of Lemma 5.4.

Thus we let $q \in A$ be such that $q_n = 0$, $mult_q(g) = D$ and $\lambda_i(q) = 0$ for all i. We shall prove the following:

Claim. There exists a finite collection of algebraic maps, $\{\pi_j : B_j \to B_q\}$ (obtained by blowing up a certain subvariety) such that for every j we have: B_j is a centered polydisk, π_j is algebraic, $\pi \circ \pi_j$ is of type $\mathcal{W}' = (n, D, R, N', s-1)$ where $\mathcal{W}'$ depends only on $\mathcal{W}$ and $B'_q \subseteq \cup_j \pi_j(B_j)$.

Together with Lemma 5.2, the claim implies part 3 of Lemma 5.4. We give now its proof.

After making the change of variables $x \mapsto x + q$, we may assume that $q = 0$. Since q is fixed, we shall simply write $B = B_q$ and $B' = B'_q$. Let $\sigma = (\sigma_1, ..., \sigma_{n-1})$ be the smallest element of $\{\alpha_i, \beta_j\}$ (which exists by assumption (8)). Then $|\sigma| \geq d!$ (since $s \geq D! \geq d!$).

Let $I \subseteq \{1, ..., n-1\}$ such that $\sum_{l\in I} \sigma_l \geq d!$ with I minimal. Thus we have

$$0 \leq \left(\sum_{l\in I} \sigma_l\right) - d! < \sigma_k \tag{5.6}$$

for all $k \in I$. Let $Z_I = \{x \in \mathbf{C}^n : x_n = 0 \text{ and } x_k = 0, k \in I\}$. We wish to consider $\pi : Y \to \mathbf{C}^n$, the blow up of $\mathbf{C}^n$ with center Z_I: In order to explain what this means, we list some properties of $\pi : Y \to \mathbf{C}^n$ which uniquely characterize it, and then explain how these properties can be used to construct $\pi : Y \to \mathbf{C}^n$.

1. Y is a complex manifold of dimension n and $p : Y \to \mathbf{C}^n$ is a holomorphic map.
2. The map $p : Y \backslash p^{-1}(Z_I) \to \mathbf{C}^n \backslash Z_I$ is a biholomorphic map.
3. Y is covered by $|I| + 1$ coordinate charts

   $$Y = \cup_{k\in I\cup\{n\}} Y^{(k)}$$

   with biholomorphic coordinate functions $\phi_k : Y^{(k)} \approx \mathbf{C}^n$: $\phi_k(y) = (y_1^{(k)}(x), ..., y_n^{(k)}(x))$.

4. The map $\pi_k = p|_{Y^{(k)}} : Y^{(k)} \to \mathbf{C}^n$ is given, in local coordinates, by the following formulas: $(x_1, ..., x_n) = \pi_k(y_1^{(k)}(x), ..., y_n^{(k)}(x))$ where, for $k = n$,

   $$x_i = y_i^{(n)} y_n^{(n)} \text{ for } i \in I; \quad x_i = y_i^{(n)} \text{ for } i \notin I .$$

   and for $k \in I$:

   $$x_n = y_n^{(k)} y_k^{(k)};\; x_l = y_l^{(k)} y_k^{(k)},\, l \in I, l \neq k;\; x_k = y_k^{(k)};\; x_l = y_l^{(k)},\, l \notin I \cup \{n\} .$$

   Now Y can be constructed as follows: It is the disjoint union of $|I| + 1$ copies of $\mathbf{C}^n$ modulo the equivalence relation forced by property 4.

Let $\tilde{Y} = p^{-1}(B)$ and, by abuse of notation, denote by π_k the restriction of p to $\tilde{Y}_k = \tilde{Y} \cap Y_k$.

Next we shall compute $f \circ (\pi \circ \pi_k)$ for each $k \in I \cup \{n\}$. We shall show that for each $k \in I$, $\pi \circ \pi_k$ has type $\mathcal{W}' = (n, D, R, N', s - 1)$ on every polydisk $E \subseteq \tilde{Y}_k$ centered at the origin, where $\mathcal{W}'$ depends only on $\mathcal{W}$. Moreover, we shall compute $f \circ (\pi \circ \pi_n)$ and shall show that for each point $q \in V = \{q \in Y_n : q_l^{(n)} = 0$ for all $l \in I\}$, there is an open centered polydisk $E \subseteq \tilde{Y}_n$ on which $\pi \circ \pi_n$ has type $\mathcal{W}' = (n, D, R, N', s - 1)$ where $\mathcal{W}'$ depends only on $\mathcal{W}$. This will complete the proof of part 2 of Lemma 5.4 since, as one easily checks, a finite collection of the E obtained in this way has the property $\cup_E\, p(E) \supset B'$.

We start with the case $k = n$: Then $\tilde{Y}_n$ has coordinates $(y_1, ..., y_n)$ (where we suppress the superscripts in order to ease notation) and in these coordinates, $x = \pi_n(y)$ is given by the following formulas:

$$x_n = y_n; \quad x_k = y_k y_n \text{ for } k \in I; \quad x_k = y_k \text{ for } k \notin I .$$

Making these substitutions in (5.2) we obtain:

$$\tilde{x}^\xi \circ \pi = \tilde{y}^\xi y_n^{\sum_{i \in I} \xi_i} \; ; \quad \lambda_i \circ \pi = y_n + u_i \tilde{y}^{\alpha_i/d!} y_n^{\sum_{l \in I} \alpha_{il}/d!} \; = \; y_n\, u_i'(y)$$

where u_i' does not vanish in an open neighborhood of the set $V = \{y : y_i = 0, i \in I\}$. Here we are using the fact that $\tilde{y}^{\alpha_i/d!}$ vanishes on V which follows from the inequality $\sum_{l \in I} \alpha_{il}/d! \geq 1$ (which in turn follows from the assumption that σ is minimal).

We also obtain:

$$(g \circ \pi)(y) \; = \; y_n^d + \sum_{j=1}^{d} u_j \tilde{y}^{\beta_j j/d!} y_n^{\sum_{l \in I} j\beta_{jl}/d!} y_n^{d-j} \; = \; y_n^d u''(y) \; .$$

where u'' does not vanish in an open neighborhood of the set V. Here we are using the fact that $\sum_{l \in I} \beta_{jl} \geq d!$ which in turn follows from the fact that σ is minimal . Thus we conclude that in a neighborhood of V, $f \circ \pi_n$ is a unit times a monomial whose degree is bounded by $|\xi| + |\xi| + |\mu| + d$.

Now we consider $f \circ (\pi \circ \pi_k)$ with $k \in I$. In this case π_k is given, in local coordinates, by the following formulas:

$$x_n = y_n y_k; \;\; x_l = y_l y_k, \; l \in I, l \neq k; \;\; x_k = y_k; \;\; x_l = y_l, \; l \notin I \cup \{n\} \; .$$

Making these substitutions in (5.2) we obtain:

$$\tilde{x}^\xi \circ \pi = \tilde{y}^\xi y_k^{\sum_{l \in I, l \neq k} \xi_l} \; ; \quad \lambda_i \circ \pi = y_n y_k + u_i y_1^{\alpha_{i1}/d!} \cdots y_k^{\sum_{l \in I} \alpha_{il}/d!} \cdots y_{n-1}^{\alpha_{i,n-1}/d!}$$

Thus $\lambda_i \circ \pi_k = y_k(y_n + u_i \tilde{y}^{\alpha_i'})$ where $\alpha_{il}' = \alpha_{il}$ for $l \neq k$ and $\alpha_{ik}' = (\sum_{l \in I} \alpha_{il} - d!)$. Finally,

$$g \circ \pi_k(y) = y_n^d y_k^d + \sum_{j=1}^d v_j y_1^{\beta_{j1} j/d!} \cdots y_k^{\sum_{l \in I} \beta_{jl} j/d!} \cdots y_{n-1}^{\beta_{j,n-1} j/d!} y_k^{d-j} y_n^{d-j} \quad =$$

$$y_k^d \left(y_n^d + \sum_{j=1}^d [v_j \tilde{y}^{\beta_j' j/d!}] y_n^{d-j} \right)$$

where $\beta_{jl}' = \beta_{jk}$ if $l \neq k$ and $\beta_{jk}' = \sum_{l \in I} \beta_{jl} - d!$. Thus we see that $f \circ (\pi \circ \pi_k)$ has a factorization similar to (5.2), satisfying conditions (1) through (9), but with α_i replaced α_i', β_j replaced by β_j', ξ replaced by ξ' where $|\xi'|$ is bounded in terms of $\mathcal{W}$, and s replaced by $s - 1$ (by virtue of (5.6)).

It remains to check that $J(\pi \circ \pi_k)$ has the factorization dictated by equation (5.2) and conditions (1) through (9) which follow, but the argument is similar to that given above, only simpler, so we omit it. Q.E.D.

We come now to the proof of Lemmas 5.1 and 5.2.

Proof of Lemma 5.1.

Claim. Let $M = \cup_{\alpha \in A} U_\alpha$ be an open cover. Then there exists $A' \subseteq A$, with A' countable and $\{U_\alpha : \alpha \in A'\}$ an open cover.

Proof of Claim. Let $\{O_i\}_{i \in \mathbf{N}}$ be a countable base for the topology of M. Choose α_i as follows: For each i, let α_i be such that $O_i \subseteq V_{\alpha_i}$, if such an α_i exists. Then the $\{V_{\alpha_i}\}$ still cover: If $p \in M$ then $p \in V_\alpha$ for some α, so $p \in O_i \in V_\alpha$ for some i, so $p \in V_{\alpha_i}$.

Claim. There exists an open covering $M = \cup_{i=1}^\infty B_i$ with $\overline{B}_i \subseteq B_{i+1}$ and $\overline{B}_i$ compact.

Proof of Claim. For every $p \in M$ let $\phi_p : U_p \to V_p \subseteq \mathbf{C}^n$ be a coordinate chart centered at p. Let $V_p^{(1)} \subseteq V_p^{(2)} \subseteq V_p^{(3)} \subseteq V_p$ be balls of different radii centered at $\phi(p)$ and let $C_p = \phi_p^{-1}(V_p^{(1)})$ and $D_p = \phi_p^{-1}(V_p^{(2)})$. Then $\{C_p\}$ is an open cover of M, $\overline{C}_p$ is compact, $\overline{D}_p$ is compact and $\overline{C}_p \subseteq D_p$. Let $\{C_{p_i}\}$ be a countable subcover of $\{C_p\}$ (which exists, by the previous claim). Now we define

B_i inductively as follows: Assume B_i has been chosen. Then $B_i \subseteq \cup_{i \leq n_i} C_i$ for some n_i. Then we let $B_{i+1} = \cup_{i \leq n_i} D_i$.

Now we prove Lemma 5.1: For every $p \in \overline{B}_i \backslash B_{i-1}$ choose $\alpha = \alpha(p)$ such that $U_\alpha \subseteq B_{i+1} \backslash \overline{B}_{i-2}$. Then the U_α cover the compact set $\overline{B}_i \backslash B_{i-1}$ and thus we can select a finite subcover: $\overline{B}_i \backslash B_{i-1} = \cup_{\alpha \in A_i} U_\alpha$, where A_i is a finite set of indices. Now let $A' = \cup A_i$. Then $M = \cup_{\alpha \in A'} U_\alpha$ is a countable, locally finite subcover.

Proof of Lemma 5.2.

Step 1. Let M be a manifold and let $\mathcal{U}$ be an l.f.r.c. open cover of M. This means that $\mathcal{U}$ is a collection of open subsets of M such that

a) $M = \cup_{U \in \mathcal{U}} U$.
b) $\overline{U} \subseteq M$ is compact for all $U \in \mathcal{U}$.
c) If $K \subseteq M$ is compact then $K \cap U = \emptyset$ for all but finitely many $U \in \mathcal{U}$.

Then we claim that there is a l.f.r.c open cover $\mathcal{V}$ such that for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $\overline{V} \subseteq U$.

To see this we fix $U \in \mathcal{U}$. Then, for every $x \in \partial U$ there exists V_x, an open set containing x, with $\overline{V}_x \subseteq U'$ for some $U' \in \mathcal{U}$. Choose an open set W_x such that $x \in W_x \subseteq \overline{W}_x \subseteq V_x$. By compactness, there exists and integer $n = n(U)$ and points $x_1, ..., x_n \in \partial U$ such that $\partial U \subseteq \cup_{i=1}^n W_{x_i}$. Let $V_o(U) = U \backslash \cup_{i=1}^n \overline{W}_{x_i}$ and $V_i(U) = V_{x_i}$ for $i = 1, 2, ..., n(U)$. Finally, let $\mathcal{V} = \{V_o(U), V_1(U), ..., V_n(U) : U \in \mathcal{U}\}$. We wish to show that $\mathcal{V}$ satisfies properties a),b) and c) above.

First, a) is clear since $U \subseteq \cup V_k(U)$ for every $U \in \mathcal{U}$. And b) is clear as well since for every k and U, $\overline{V_k(U)} \subseteq U'$ for some U' and $\overline{U}'$ is assumed to be compact.

Thus we need to verify c). We note that for $U \in \mathcal{U}$ fixed, that

$$\{U' \in \mathcal{U} : U \cap U' \neq \emptyset\} \text{ is finite.} \tag{5.7}$$

In fact, by the locally finite property for $\mathcal{U}$, $\{U' \in \mathcal{U} : \overline{U} \cap U' \neq \emptyset\}$ is finite. Next we observe that for $U \in \mathcal{U}$ fixed,

$$\{V \in \mathcal{V} : V \subseteq U\} \text{ is finite.} \tag{5.8}$$

This follows since every V is of the form $V = V_k(U')$ for some $U' \in \mathcal{U}$, $1 \leq k \leq n(U')$. If $V \subseteq U$ then $V_k(U') \subseteq U$ which implies that $U \cap U' \neq \emptyset$. Hence (5.8) follows from (5.7).

Now let $K \subseteq M$ be compact. We know that

$$\{U \in \mathcal{U} : U \cap K \neq \emptyset\} \text{ is finite} \tag{5.9}$$

by virtue of the locally finite property of $\mathcal{U}$. Now if $V \in \mathcal{V}$ and $V \cap K \neq \emptyset$, then there exists $U \supseteq V$ with $U \in \mathcal{U}$ such that $U \cap K \neq \emptyset$. Combining (5.8) and

(5.8) we see that $\{V \in \mathcal{V} : V \cap K \neq \emptyset\}$ is finite. This completes the verification of (c).

Step 2. Suppose that $\pi_i(M_i)$ is subordinate to $\mathcal{U} = \{U_i\}_{i\in I}$. Choose $\mathcal{V}$ as in step one and let $V \in \mathcal{V}$. Then $\overline{V}$ compact implies $\overline{V} = \cup\, \pi_i(K_i(V))$ where $K_i(V) \subseteq M_i$ is compact and the union is finite. Fix i. Then we claim that $K_i(V) \neq \emptyset$ for all but finitely many $V \in \mathcal{V}$: Indeed, if $K_i(V) \neq \emptyset$ then $\pi_i(K_i(V)) \neq \emptyset$ which implies that $\overline{V} \cap U_i \neq \emptyset$ and thus $V \cap U_i \neq \emptyset$ which implies $V \cap \overline{U}_i \neq \emptyset$, which occurs for at most a finite number of V. Let $C_i = \cup_{V\in\mathcal{V}} K_i(V)$ with $K_i(V) \subseteq M_i$ compact. This is a finite union so $C_i \subseteq M_i$ is compact. Hence, for each i there is a finite set $J = J(i)$ and compact set $K_{ij} \subseteq M_{ij}$ for $i \in I$ and $j \in J(i)$ such that $C_i = \cup_{j\in J(i)} \pi_{ij}(K_{ij})$.

Step 3. We claim that $\{\pi_i \circ \pi_{ij} : M_{ij} \to M_i : i \in I,\, j \in J(i)\}$ is an l.f.r.c. cover of M: Note that $\pi_i \circ \pi_{ij}(M_{ij}) \subseteq \pi_i(M_i) \subseteq U_i$. Thus we let $\mathcal{U}^* = \{U_{ij} : i \in I,\, j \in J(i)\}$ be the family of open sets defined by $U_{ij} = U_i$. Then $\mathcal{U}^*$ is an l.f.r.c cover of M by open sets (since $\mathcal{U}$ is and since $J(i)$ is finite) and $\pi_i \circ \pi_{ij}(M_{ij})$ is subordinate to $\mathcal{U}^*$.

Now let $K \subseteq M$ be compact. Then $K = \cup_{V\in\mathcal{V}}(K \cap \overline{V})$ is a finite union. On the other hand, $\overline{V} = \cup\, \pi_i(K_i(V))$ so $\overline{V} \cap K = \cup_i\, \pi_i(K_i(V)) \cap K = \cup_i \pi_i(\tilde{K}_i(V))$ for some compact $\tilde{K}_i(V) \subseteq K_i(V) \subseteq C_i$. But $C_i = \cup_{j\in J(i)}\, \pi_i(K_{ij})$ which implies $\tilde{K}_{ij} = \cup\, \pi_{ij}(\tilde{K}_{ij}(V))$ for some $\tilde{K}_{ij}(V) \subseteq M_{ij}$. Thus $V \cap \overline{K} = \cup_i \pi_i(\tilde{K}_i(V)) = \cup_i \cup_{j\in J(i)}\ \pi_i\pi_{ij}(\tilde{K}_{ij}(V))$ and thus $K = \cup_V \cup_i \cup_{j\in J(i)}\ \pi_i\pi_{ij}(\tilde{K}_{ij}(V))$ with $\tilde{K}_{ij}(V)) \subseteq M_{ij}$ compact.

6. Examples of global instability

Let $\delta > 0$ and $f \in \mathbf{C}[z_1, ..., z_n; t]$ be a polynomial in $n + 1$ variables. For $B \subseteq \mathbf{C}^n$ and $t \in \mathbf{C}$ define

$$I(t) \;=\; \int_B |f(z,t)|^{-\delta}\, dV_1 \cdots dV_n \tag{6.1}$$

The integral defined by (6.1) is locally stable, that is, if B is a compact domain, we know, from [PS1] or [DK], that $I(t)$ is continuous. In this section we provide examples showing that global stability fails.

One-dimensional examples

Consider the following integrals

$$I(t) = \int_{\mathbf{C}} \frac{1}{|(1-tz)^M \prod_{j=1}^{N}(z-\alpha_j)|^\delta} dV.$$

with $\alpha_j \neq \alpha_k$ for $j \neq k$, $0 < \delta < 2$, $\delta N > 2$, and $M\delta > 2$. When $t = 0$, the only finite zeroes of the denominator are simple zeroes at α_j, and hence the integral converges there since $\delta < 2$. At $z = \infty$, the integral converges since $\delta N > 2$. Thus $I(0)$ is finite. On the other hand, for $t \neq 0$, the denominator has an additional finite zero at $z = t^{-1}$, and $I(t) = \infty$ since $M\delta > 2$. Thus we have

$$I(t) < \infty \text{ if and only if } t = 0\,. \tag{6.2}$$

We observe however that more restricted forms of stability may hold in one dimension. For example, consider a holomorphic family $f(z,t)$ of the form

$$f(z,t) = z^N + a_1(t)z^{N-1} + \cdots + a_N(t)$$

where $t \in \mathbf{C}^K$ is a small parameter, and $a_j(t)$, $1 \leq j \leq N$, are holomorphic functions of t. Then it is easy to see that

$$\int_{\mathbf{C}} |f(z,0)|^{-\delta} dV < \infty \Rightarrow \int_{\mathbf{C}} |f(z,t)|^{-\delta} dV < \infty$$

for $|t|$ small enough. Indeed, consider the region of integration $|z| > A$, and write

$$\int_{|z|>A} |f(z,t)|^{-\delta} dV(z) = \int_{|w|<A^{-1}} \frac{|w|^{N\delta-4}}{|1 + a_1(t)w + \cdots + a_N(t)w^N|^{\delta}} dV(w).$$

Let P be the largest index j with $a_j(0) \neq 0$. Then the Weierstrass Preparation Theorem with parameters (see e.g. [PSS1], p. 543) shows that we may write

$$1 + a_1(t)w + \cdots + a_N(t)w^N = u(w,t)\left(w^P + \sum_{k=0}^{P-1} b_{P-k}(t)w^k\right)$$

for $|t|, |w| < \epsilon$, where $u(w,t)$, $b_{P-k}(t)$ are holomorphic functions, $|u(w,t)|$ is bounded away from 0, and ϵ is some positive number. Shrinking ϵ further, we may also assume that the polynomial $w^P + \sum_{k=0}^{P-1} b_{P-k}(t)w^k$ does not vanish for $w = 0$. Choose now A with $A > \epsilon^{-1}$. Then the finiteness of the integral of $|f(z,0)|^{-\delta}$ over the region $|z| > A^{-1}$ means $N\delta > 2$ and that the zeroes of the polynomial $w^P + \sum_{k=0}^{P-1} b_{P-k}(0)w^k$ have maximum multiplicity $m(0)$ with $m(0)\delta < 2$. Since multiplicities are non-increasing under perturbations, it follows that the maximum multiplicity $m(t)$ of $w^P + \sum_{k=0}^{P-1} b_{P-k}(t)w^k$ still satisfies $m(t)\delta < 2$. This implies that $\int_{|z|>A} |f(z,t)|^{-\delta} dV(z) < \infty$. Since $\int_{|z|\leq A} |f(z,t)|^{-\delta} dV(z) < \infty$ for $|t|$ small enough by the local result on stability, the desired result follows.

Two-dimensional examples

In two-dimensions, perturbations which may appear harmless in view of the previous one-dimensional result can still cause instability. Consider the following integrals

$$\tilde{I}(t) \;=\; \int_{\mathbf{C}^2} \frac{1}{|z_1 - t|^4 + |z_1 z_2 - 1|^4} \cdot \frac{dV_1 dV_2}{1 + |z_1|^6 + |z_2|^6 + |z_1 z_2|^6} \tag{6.3}$$

Claim. $\tilde{I}(t) < \infty$ if and only if $t = 0$.

Proof of Claim. Assume first that $t \neq 0$. Then the denominator of the integrand in (6.3) vanishes at the point $(t, 1/t)$. The fact that $\tilde{I}(t) = \infty$ thus follows from the equation

$$\int_B (|x_1| + |x_2| + |x_3| + |x_4|)^{-4}\, dx_1 dx_2 dx_3 dx_4 \;=\; \infty$$

where $B \subseteq \mathbf{R}^4$ is any open set containing the origin.

Now assume that $t = 0$. For $J \subseteq \{1, 2\}$ we let

$$D_J = \{z = (z_1, z_2) \in \mathbf{C}^2 : |z_j| > 1 \text{ if and only if } j \in J\},$$

and we let $D = D_\emptyset$. Then $\mathbf{C}^2 = \cup_J D_J$ is a disjoint union, so, corresponding to this decomposition, we can write $\tilde{I}(0) = \sum_J \tilde{I}_J$. We must now verify that $\tilde{I}_J < \infty$ for all J.

When $J = \emptyset$ we clearly have $\tilde{I}_J < \infty$ since the integral of a continuous function of a compact domain is finite.

When $J = \{1\}$ then, upon making the change of variables $z_1 \mapsto z_1^{-1}$, we obtain:

$$\tilde{I}_{\{1\}} \;=\; \int_D \frac{|z_1^8|}{1 + |z_1 - z_2|^4}\, d\mu \tag{6.4}$$

where $d\mu = (1 + |z_1|^6 + |z_2|^6 + |z_1 z_2|^6)^{-1} dV_1 dV_2$. We see that $\tilde{I}_{\{1\}} < \infty$ since the denominator of (6.4) never vanishes.

When $J = \{2\}$ then, upon making the change of variables $z_2 \mapsto z_2^{-1}$, we obtain:

$$\tilde{I}_{\{2\}} \;=\; \int_D \frac{|z_2^8|}{|z_1 z_2|^4 + |z_2 - z_1|^4}\, d\mu$$

Now the identity $z_2^2 = z_1 z_2 + z_2(z_2 - z_1)$ implies that $|z_2|^8 \leq 16(|z_1 z_2|^4 + |z_1 - z_2|^4)$ provided $(z_1, z_2) \in D$. This shows that $\tilde{I}_{\{2\}}$ is finite.

Finally, when $J = \{1, 2\}$ then making the change of variables $z_1 \mapsto z_1^{-1}$ and $z_2 \mapsto z_2^{-1}$, we obtain:

$$\tilde{I}_{\{1,2\}} \;=\; \int_D \frac{|z_1 z_2|^8}{|z_2|^4 + |1 - z_1 z_2|^4}\, d\mu$$

which is finite since the denominator never vanishes. This finishes the proof of the Claim.

The preceding example can be considered as an example of instability of the finiteness of integrals of the form $\int_{\mathbf{C}} ||f(z,t)||^{-\delta} dV(z)$, where $f(z,t)$ is a vector-valued polynomial. To get scalar examples (and integrals of the form $I(t)$ rather than $\tilde{I}(t)$), we proceed as follows.

Observe that the product of the denominators appearing in (6.3) is a sum of eight terms. Letting $Q_1, ..., Q_8$ be the squares of those eight terms, we obtain

$$\tilde{I}(t) \;=\; \int_{\mathbf{C}^2} \frac{1}{\sum_{k=1}^{8} |Q_k(z,t)|^{1/2}}\, dV_1 dV_2$$

By Lemma 4.9 of [PS1], we can find $\zeta_1, ..., \zeta_8 \in \mathbf{C}$ such that $|\zeta_i| = 1$ for all i with the property:

$$\int_{\mathbf{C}^2} \frac{1}{|\sum_{k=1}^{8} \zeta_k Q_k(z,0)|^{1/2}}\, dV_1 dV_2 \;<\; \infty\;. \tag{6.5}$$

Thus we let $f(z,t) = \sum_{k=1}^{8} \zeta_k Q_k(z,t)$ and

$$I(t) \;=\; \int_{\mathbf{C}^2} |f(z,t)|^{-1/2}\, dV_1 dV_2\;.$$

Then (6.5) together with the Claim implies that (6.2) is satisfied.

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