

The Monge-Ampère operator and geodesics in the space of Kähler potentials*

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1. Introduction

Let X be an n -dimensional compact complex manifold, $L \rightarrow X$ a positive holomorphic line bundle, and \mathcal{H} the space of positively curved hermitian metrics on L . The purpose of this article is to prove that geodesics in the infinite-dimensional symmetric space \mathcal{H} can be uniformly approximated by geodesics in the finite-dimensional symmetric spaces $\mathcal{H}_k = GL(N_k + 1, \mathbf{C})/U(N_k + 1, \mathbf{C})$, where $N_k + 1 = \dim(H^0(L^k))$. Thus the $\mathcal{H}_k \subseteq \mathcal{H}$ are becoming flat as $k \rightarrow \infty$.

The motivation for this work comes from Donaldson's far reaching program [10, 11] relating the geometry of \mathcal{H} to the existence and uniqueness of constant scalar curvature Kähler metrics. As advocated by S.T. Yau over the years, \mathcal{H} should be approximated by \mathcal{H}_k , and the properties of this approximation should be closely reflected in many basic questions of Kähler geometry. In particular, the condition of "stability" is one which concerns the growth of energy functionals along the geodesics of \mathcal{H}_k . On the other hand, the existence and uniqueness of metrics of constant scalar curvature concerns the growth of energy functionals along the geodesics of \mathcal{H} (see [12, 13, 27, 28, 30]). Thus a good understanding of the relationship between these different types of geodesics is desirable.

In order to state the precise theorem, we need some notation: Let $h : L \rightarrow [0, \infty)$ be a smooth hermitian metric. If $s \in L$ we write $h(s) = |s|_h$ and for $k > 0$, we denote by h^k the induced metric on L^k . The curvature of h is the $(1, 1)$ form on X defined locally by the formula $R(h) = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log |s(z)|_h^2$, where $s(z)$ is a local, nowhere vanishing holomorphic section. In particular,

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$R(h^k) = kR(h)$. Let

$$\mathcal{H} = \{h : L \rightarrow [0, \infty) : h \text{ is smooth metric on } L \text{ such that } R(h) > 0\}.$$

If we fix $h_0 \in \mathcal{H}$ and let $\omega_0 = R(h_0)$ we have a natural isomorphism

$$\mathcal{H} = \left\{ \phi \in C^\infty(X) : \omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\phi > 0 \right\},$$

where ϕ is identified with $h = h_0 e^{-\phi}$ so that $R(h) = \omega_\phi$. Then \mathcal{H} is an infinite dimensional manifold whose tangent space $T_\phi \mathcal{H}$ at $\phi \in \mathcal{H}$ is naturally identified with $C^\infty(X)$.

Now let $\phi \in \mathcal{H}$ and $\psi \in C^\infty(X)$ and define a metric on \mathcal{H} by

$$\|\psi\|_\phi^2 = \int_X |\psi|^2 \omega_\phi^n. \tag{1.1}$$

Donaldson [11], Mabuchi [20] and Semmes [25] have shown that (1.1) defines a Riemannian metric which makes \mathcal{H} into an infinite dimensional negatively curved symmetric space. Furthermore, the geodesics of \mathcal{H} in this metric are the paths ϕ_t which satisfy the partial differential equation

$$\ddot{\phi} - |\dot{\phi}|_{\omega_\phi}^2 = 0. \tag{1.2}$$

The space \mathcal{H} contains a canonical family of finite-dimensional negatively curved symmetric spaces \mathcal{H}_k which are defined as follows: For $k \gg 0$ and for $\underline{s} = (s_0, \dots, s_{N_k})$ an ordered basis of $H^0(L^k)$, let

$$\iota_{\underline{s}} : X \hookrightarrow \mathbf{P}^N$$

be the Kodaira embedding given by $z \mapsto (s_0(z), \dots, s_{N_k}(z))$. Then we have a canonical isomorphism $L^k = \iota_{\underline{s}}^* O(1)$. Let h_{FS} be the Fubini-Study metric on $O(1) \rightarrow \mathbf{P}^N$ and let

$$h_{\underline{s}} = (\iota_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{N_k} |s_j|_{h_0^k}^2\right)^{1/k}}. \tag{1.3}$$

Note that the right side of (1.3) is independent of the choice of h_0 . In particular

$$\sum_{j=0}^{N_k} |s_j|_{h_{\underline{s}}^k}^2 = 1. \tag{1.4}$$

Let

$$\mathcal{H}_k = \{h_{\underline{s}} : \underline{s} \text{ a basis of } H^0(L^k)\} \subseteq \mathcal{H}.$$

Then $\mathcal{H}_k = GL(N_k + 1)/U(N_k + 1)$ is a finite-dimensional negatively curved symmetric space (non-isometrically) imbedded in \mathcal{H} . If $k|l$ then $\mathcal{H}_k \subseteq \overline{\mathcal{H}_l}$. Hence, if $k_1, k_2 > 0$ there exists l such that $\mathcal{H}_{k_1} \subseteq \overline{\mathcal{H}_l}$ and $\mathcal{H}_{k_2} \subseteq \overline{\mathcal{H}_l}$. It is well known that the \mathcal{H}_k are topologically dense in \mathcal{H} : If $h \in \mathcal{H}$ then there exists $h(k) \in \mathcal{H}_k$ such that $h(k) \rightarrow h$ in the C^∞ topology. This follows from the Tian-Yau-Zelditch theorem on the density of states (Yau [29], Tian [26] and Zelditch [31]; see also Catlin [4] for corresponding results for the Bergman kernel). In fact, if $h \in \mathcal{H}$, then there is a canonical choice of the approximating sequence $h(k)$: Let \underline{s} be a basis of $H^0(L^k)$ which is orthonormal with respect to the metrics h . In other words,

$$\int_X (s_i, s_j)_{h^k} \omega^n = \delta_{ij} \quad \text{where } \omega = R(h). \tag{1.5}$$

Note that \underline{s} is not unique: if \underline{s} is orthonormal, so is $u \underline{s}$, for any $u \in U(N_k + 1)$. Define $\rho_k(h) = \rho_k(\omega) = \sum_j |s_j|_{h^k}^2$. Then Theorem 1 of [31], which is the C^∞ version of the C^2 approximation result first established in [26], says that for h fixed, we have a C^∞ asymptotic expansion as $k \rightarrow \infty$:

$$\rho_k(\omega) \sim k^n + A_1(\omega)k^{n-1} + A_2(\omega)k^{n-1} + \dots \tag{1.6}$$

Here the $A_j(\omega)$ are smooth functions on X defined locally by ω which can be computed in terms of the curvature of ω by the work of Lu [19]. Let $\hat{\underline{s}} = k^{-n/2} \underline{s}$ and $h(k) = h_{\hat{\underline{s}}}$. Then (1.3) and (1.6) imply $\frac{h}{h(k)} = [k^{-n} \rho_k(\omega)]^{1/k} = [1 + O(\frac{1}{k})]^{1/k} = 1 + O(\frac{1}{k^2})$. In particular, (1.3) and (1.6) show that for each $r > 0$

$$\begin{aligned} \left\| \frac{h(k)}{h} - 1 \right\| &= O\left(\frac{1}{k^2}\right), & \|\omega(k) - \omega\| &= O\left(\frac{1}{k^2}\right), \\ \|\phi(k) - \phi\| &= O\left(\frac{1}{k^2}\right), \end{aligned} \tag{1.7}$$

where the norms are all taken with respect to $C^r(\omega_0)$. Here, as before, $\omega = R(h)$, $\omega(k) = R(h(k))$, $h = h_0 e^{-\phi}$ and $h(k) = h_0 e^{-\phi(k)}$.

Now let $h_0, h_1 \in \mathcal{H}$. It is known by the work of Chen [6] (see also more recent progress in Donaldson [14] and Chen-Tian [7]) that there is a unique $C^{1,1}$ geodesic $h : [0, 1] \rightarrow \mathcal{H}$ joining h_0 to h_1 . As discussed in Donaldson [11] the optimal regularity properties of this geodesic are of considerable interest. Our main theorem says that $h_t = h_0 e^{-\phi(t)}$ is a uniform limit of the smooth geodesics in $\mathcal{H}_k = GL(N_k + 1)/U(N_k + 1)$ which join $h_0(k) = h_{\hat{\underline{s}}^{(0)}}$ to $h_1(k) = h_{\hat{\underline{s}}^{(1)}}$.

To be precise, let $\sigma \in GL(N_k + 1)$ be the change of basis matrix defined by $\sigma \cdot \hat{\underline{s}}^{(0)} = \hat{\underline{s}}^{(1)}$. Without loss of generality, we may assume that σ is diagonal with entries $e^{\lambda_0}, \dots, e^{\lambda_N}$ for some $\lambda_j \in \mathbf{R}$. Let $\hat{\underline{s}}^{(t)} = \sigma^t \cdot \hat{\underline{s}}^{(0)}$ where σ^t is diagonal with entries $e^{\lambda_j t}$. Define

$$h(t; k) = h_{\hat{\underline{s}}^{(t)}} = h_0 e^{-\phi(t; k)}. \tag{1.8}$$

Then $h(t; k)$ is the smooth geodesic in $GL(N_k + 1)/U(N_k + 1)$ joining $h_0(k)$ to $h_1(k)$. Explicitly, using (1.3), we can also write

$$\phi(t; k) = \frac{1}{k} \log \left(\sum_{j=0}^N e^{2\lambda_j t} |\hat{s}_j^{(0)}|_{h_0^k}^2 \right). \tag{1.9}$$

Theorem 1 *Let $h_0, h_1 \in \mathcal{H}$ and $h_t = h_0 e^{-\phi_t}$ the unique $C^{1,1}$ geodesic joining h_0 to h_1 . Then*

$$\phi_t = \lim_{l \rightarrow \infty} [\sup_{k \geq l} \phi(t; k)]^* \quad \text{uniformly as } l \rightarrow \infty, \tag{1.10}$$

where, for any bounded function $u : X \times [0, 1] \rightarrow \mathbf{R}$, we define the upper envelope u^* of u by $u^*(\zeta_0) = \lim_{\epsilon \rightarrow 0} \sup_{|\zeta - \zeta_0| < \epsilon} u(\zeta)$.

Remark 1 The proof will show that

$$\phi_t = \lim_{l \rightarrow \infty} [\sup_{k \geq l} \phi(t; k)] \quad \text{almost everywhere.} \tag{1.11}$$

Remark 2 We note that the upper envelope u^* is independent of the choice of coordinate systems defining the balls $|\zeta - \zeta_0| < \epsilon$. It is the smallest upper semicontinuous function which is greater than or equal to u . If u_k is a sequence of plurisubharmonic functions which are locally uniformly bounded, then $[\sup u_k]^*$ is plurisubharmonic and equal to $\sup u_k$ almost everywhere. Similarly, we also define for later use the lower envelope u_* of a bounded function u by $u_*(\zeta_0) = \lim_{\epsilon \rightarrow 0} \inf_{|\zeta - \zeta_0| < \epsilon} u(\zeta)$. The function u_* is the largest lower semi-continuous function which is less than or equal to u .

To prove the theorem we first apply the observation of Donaldson [11], Mabuchi [20] and Semmes [25], which shows that solving the geodesic equation on \mathcal{H} is equivalent to solving the degenerate Monge-Ampère equation

$$\Omega_\Phi^{n+1} = 0 \tag{1.12}$$

on the manifold $\bar{M} = X \times A$, where A is the annulus $A = \{w \in \mathbf{C} : 1 \leq |w| \leq e\}$, the values of Φ are prescribed on the boundary of \bar{M} by smooth rotationally symmetric data, and $\Omega_\Phi|_{X \times \{t\}}$ is positive for every t . Here we are writing $\Omega_\Phi = \Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi$ where $\Omega_0 = \pi_1^* \omega_0$ and Φ is a smooth on \bar{M} . We then use the Tian-Yau-Zelditch theorem to prove that the \mathcal{H}_k geodesics are, in a certain sense, approximate solutions to (1.12). A key step is then proving that the limit of the sequence of approximate solutions is in fact a weak solution of the Monge-Ampère equation and that the weak solution thus obtained is unique. This is accomplished using the methods of pluripotential theory which were introduced and systematically developed by Bedford-Taylor in their fundamental work [1] and [2]. In

recent years, this subject has been extended by Demailly [9], Klimek [16], Blocki [3], and others. The uniqueness implies that the limit coincides with the $C^{1,1}$ solution ϕ_t . Now the uniqueness theorem for bounded open sets in \mathbf{C}^n , which is due to Bedford-Taylor, requires smoothing techniques for rough solutions which do not generalize in a simple way to manifolds. It is quite possible that the regularization methods of Demailly, which have recently been successfully applied by Guedj and Zeriahi [15] in the setting of compact manifolds, will also apply in our setting. However, we use a different approach which exploits the particular structure of the weak solutions in our case. A particularly important ingredient for us is the existence of a vector field Y , transversal to the boundary of \bar{M} , with

$$|Y(\Phi_k)| \leq C, \quad (1.13)$$

where Φ_k is a sequence of approximate solutions to the Dirichlet problem for the Monge-Ampère equation (c.f. Theorem 3 and the proof of Theorem 6 below). For the application to Theorem 1, this hypothesis is a consequence of the a priori estimate

$$|\dot{\phi}(t, k)| \leq C, \quad (1.14)$$

which is the one global estimate for the derivatives of $\phi(t; k)$ that we can actually obtain.

One approach to the question raised in Donaldson [11] on the optimal regularity of geodesics is the establishment of a priori estimates on the smooth approximate solutions $\phi(t; k)$. As we just noted, the uniform C^0 estimates for $\phi(t; k)$ and for $\dot{\phi}(t; k)$ do hold. But extending these to higher derivative estimates seems rather hard, and at the same time, quite intriguing: For example, the estimate which is needed for $\ddot{\phi}(t; k)$ can be carried out in certain special cases, and appears to be related to the central limit theorem in probability theory. In the last section, we shall make some remarks along these lines.

2. The volume estimate

In this section, we establish the basic properties of the functions $\phi(t; k)$ introduced in (1.9). The properties of the functions $\phi(t; k)$ themselves are summarized in Lemma 1 below, while the properties of their corresponding Monge-Ampère measures are given in Theorem 2.

- A first key ingredient is the following very general volume formula. Let (X, ω_0) be a compact Kähler manifold of dimension n , $A = \{w \in \mathbf{C} : 1 \leq |w| \leq e\}$, $\bar{M} = X \times A$, $\pi_1 : \bar{M} \rightarrow X$ be the projection on the first factor, and $\Omega_0 = \pi_1^* \omega_0$. Then Ω_0 is a closed positive $(1, 1)$ -form on \bar{M} such that $\Omega_0^{n+1} = 0$.

Let $\phi : [0, 1] \rightarrow \phi(t) \in \mathcal{H} = \{\phi \in C^\infty(X) : \omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi\}$ be a smooth path joining $\phi(0)$ to $\phi(1)$ and define $\Phi : \bar{M} \rightarrow \mathbf{R}$ by

$$\Phi(z, w) = \phi(t)(z) \quad \text{where } t = \log |w| \quad (2.1)$$

and the corresponding (1, 1)-form Ω_Φ by

$$\Omega_\Phi = \Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi. \quad (2.2)$$

In local coordinates z^i for X and $v = \log w$ for A , if we identify a Kähler form $\omega_0 = \frac{\sqrt{-1}}{2} \sum_{ij} g_{\bar{j}i}^0 dz^i \wedge d\bar{z}^j$ with the hermitian matrix $\{g_{\bar{j}i}^0\}$, then we can write

$$\Omega_\Phi = \begin{pmatrix} g_{\bar{j}i}^0 + \partial_i \partial_{\bar{j}} \phi & \frac{1}{2} B \\ \frac{1}{2} B^* & \frac{1}{4} \ddot{\phi} \end{pmatrix}$$

where B is the column vector

$$B = \begin{pmatrix} \frac{\partial \phi}{\partial \bar{z}_1} \\ \frac{\partial \phi}{\partial \bar{z}_2} \\ \vdots \\ \frac{\partial \phi}{\partial \bar{z}_n} \end{pmatrix}$$

and B^* is the conjugate transpose of B . It follows that

$$\Omega_\Phi^{n+1} = \frac{1}{4} (\ddot{\phi} - |\partial \dot{\phi}|_{\omega_\phi}^2) \omega_\phi^n \wedge \left(\frac{\sqrt{-1}}{2} dv \wedge d\bar{v} \right), \quad (2.3)$$

and the condition $\Omega_\Phi^{n+1} = 0$ is equivalent to the geodesic equation $\ddot{\phi} - |\partial \dot{\phi}|_{\omega_\phi}^2 = 0$. This is a key observation which was pointed out in [11], [20] and [25]. Now

$$\int_{X \times A} \Omega_\Phi^{n+1} = \int_0^1 \int_X (\ddot{\phi} - |\partial \dot{\phi}|^2) \omega_\phi^n dt = \int_0^1 \frac{d}{dt} \left(\int_X \dot{\phi} \omega_\phi^n \right) dt \quad (2.4)$$

and we obtain the desired volume formula:

$$\int_{X \times A} \Omega_\Phi^{n+1} = \int_X \dot{\phi}(1) \omega_{\phi(1)}^n - \int_X \dot{\phi}(0) \omega_{\phi(0)}^n. \quad (2.5)$$

Note that (2.5) is only a true volume if $\Omega_\Phi \geq 0$, but that (2.5) is valid even in the absence of this hypothesis.

- It is perhaps noteworthy that $\frac{1}{V} \int_X \dot{\phi} \omega_\phi^n = \dot{F}_{\omega_{\phi_0}}^0$, where $V = \int_X \omega_\phi^n$, and $F_{\omega_{\phi_0}}^0(\phi)$ is precisely the functional whose critical points (with the

constraint $\frac{1}{V} \int_X e^{f_0 - \phi} \omega_0^n = 1$, and f_0 is a fixed function satisfying $Ric(\omega_0) - \mu\omega_0 = \frac{\sqrt{-1}}{2} \partial\bar{\partial}(f_0)$, give Kähler-Einstein metrics. Thus the preceding relation can be rewritten as

$$\frac{1}{V} \int_{X \times A} \Omega_{\Phi}^{n+1} = \int_0^1 \dot{F}_{\omega_{\phi_0}}^0 dt = \dot{F}_{\omega_{\phi_0}}^0(\phi(1)) - \dot{F}_{\omega_{\phi_0}}^0(\phi(0)). \quad (2.6)$$

The asymptotic behavior of $F_{\omega_{\phi_0}}^0$ along geodesics in \mathcal{H}_k is known to be closely related to Chow-Mumford stability (see [32], and also [21], [22], [24]). The simple formula (2.6) raises the intriguing possibility that the behavior of the Monge-Ampère operator Ω_{Φ}^{n+1} and geodesics in \mathcal{H} can be linked even more directly to stability and constant scalar curvature Kähler metrics.

- Fix now two potentials h_0 and h_1 in \mathcal{H} , and let $\underline{s}^{(0)}, \underline{s}^{(1)}$ be bases for $H^0(L^k)$ which are orthonormal with respect to the metrics induced by h_0 and h_1 . As in the Introduction, let $\sigma \in GL(N_k + 1)$ be the corresponding change of bases, $\sigma \cdot \underline{s}^{(0)} = \underline{s}^{(1)}$. We may assume that σ is diagonal with eigenvalues $e^{\lambda_0}, \dots, e^{\lambda_{N_k}}$. We consider the functions $\phi(t; k)$ defined as in (1.9).

Lemma 1 *Let $\Omega_0 = \pi_1^* \omega_0$ where $\omega_0 = R(h_0)$.*

- (a) *For each k , let $\Phi(k)$ be the extension of $\phi(t; k)$ to $\bar{M} = X \times A$ as in (2.1), and let $\Omega_{\Phi(k)}$ is the corresponding $(1, 1)$ -form as in (2.2). Then $\Omega_{\Phi(k)}$ is a smooth positive $(1, 1)$ -form,*

$$\Omega_{\Phi(k)} \geq 0. \quad (2.7)$$

In particular, the $(n + 1, n + 1)$ -form $\Omega_{\Phi(k)}^{n+1}$ form is a positive smooth measure on \bar{M} ;

- (b) *There is a constant $C > 0$ which does not depend on k such that*

$$C^{-1}k \leq \max_{0 \leq j \leq N} |\lambda_j| \leq Ck; \quad (2.8)$$

- (c) *With the same constant $C > 0$, we also have*

$$|\phi(t; k)| + |\dot{\phi}(t; k)| \leq 4C. \quad (2.9)$$

Proof. Locally on \bar{M} , we have $e^{2\lambda_j t} = |w^{\lambda_j}|^2$, where $w^{\lambda_j} = e^{\lambda_j(t+is)} \in A$ is a holomorphic function. Since the logarithms of sums of squares of absolute values of holomorphic functions are plurisubharmonic in the usual sense on \mathbb{C}^{n+1} , and since we can write

$$\begin{aligned} \Omega_{\Phi(k)} &= \Omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \left(\frac{1}{k} \log \sum_{j=0}^{N_k} e^{2\lambda_j t} |s_j(z)|_{h_0^k}^2 \right) \\ &= \frac{\sqrt{-1}}{2k} \partial\bar{\partial} \log \sum_{j=0}^{N_k} |w^{\lambda_j} s_j(z)|^2, \end{aligned} \quad (2.10)$$

where $|s_j(z)|$ is just the absolute value of $\hat{s}_j(z)$ in a local trivialization, the positivity of $\Omega_{\Phi(k)}$ follows. Here we have simplified the notation by denoting $s_j^{(0)}$ just by s_j .

We turn next to the proof of (b). Let $\phi = \log \frac{h_1}{h_0}$, and order the eigenvalues λ_j so that $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_N$. Clearly,

$$\frac{2}{k} \lambda_N \leq \frac{1}{k} \log \frac{\sum_{j=0}^{N_k} e^{2\lambda_j} |s_j(z)|^2}{\sum_{j=0}^{N_k} |s_j(z)|^2} \leq \frac{2}{k} \lambda_0. \tag{2.11}$$

By the Tian-Yau-Zelditch theorem, the expression in the middle tends to ϕ as $k \rightarrow \infty$. Thus, if we set $C_1 = \sup \phi$ and $C_2 = -\inf \phi$, we have $\lambda_0 \geq C_1 k/4$ and $\lambda_N \leq -C_2 k/4$, for k large enough.

To get inequalities in the opposite direction, we note that for each j ,

$$\int |s_j|_{h_0^k}^2 \frac{\omega_0^n}{n!} = 1, \quad \int \frac{1}{N_k} \sum_{l=0}^N |s_l|_{h_0^k}^2 \frac{\omega_0^n}{n!} = 1. \tag{2.12}$$

This implies that for some $x \in X$, we have $|s_j|^2 \geq \frac{1}{N_k} \sum_{l=0}^N |s_l|^2$. Choosing $j = 0$, we obtain for k large,

$$\frac{1}{k} \log \frac{e^{\lambda_0}}{N_k} \leq \frac{1}{k} \log \frac{e^{\lambda_0} |s_0(x)|^2}{|\underline{s}(x)|^2} \leq \frac{1}{k} \log \frac{\sum e^{\lambda_l} |s_l(x)|^2}{\sum |s_l(x)|^2} \leq 2C_1. \tag{2.13}$$

This shows that $\lambda_0 \leq 2C_1 k + \log N_k$. But $N_k \sim k^n$. Thus we conclude

$$\frac{C_1}{2} k \leq \lambda_0 \leq 3C_1 k, \quad k \gg 1. \tag{2.14}$$

To get the bound on λ_{N_k} , interchange the roles of h_0 and h_1 , which changes ϕ to $-\phi$ and σ to σ^{-1} . Thus we have

$$\frac{C_2}{2} k \leq -\lambda_{N_k} \leq 3C_2 k, \tag{2.15}$$

and (b) is proved.

The first inequality in (c), namely that $|\phi(t; k)|$ is uniformly bounded, follows immediately from the bounds in (b) for $|\lambda_j|$. To get the second inequality, we write

$$\dot{\phi}(t; k) = \frac{1}{k} \frac{\sum_{j=0}^{N_k} 2\lambda_j e^{2\lambda_j t} |s_j(z)|^2}{\sum_{j=0}^{N_k} e^{2\lambda_j t} |s_j(z)|^2} \tag{2.16}$$

and hence

$$|\dot{\phi}(t; k)| \leq 2 \frac{\max |\lambda_j|}{k}. \tag{2.17}$$

Applying (b) again, we find that $|\dot{\phi}|$ is uniformly bounded. □

- We can now state and prove the following theorem, which is a key step in the construction of a generalized solution of the Monge-Ampère equation $\Omega_{\phi}^{n+1} = 0$:

Theorem 2 *The volume of $X \times A$ approaches zero as k tends to infinity. More precisely,*

$$0 \leq \int_{X \times A} \Omega_{\Phi(k)}^{n+1} \leq \frac{C}{k} \tag{2.18}$$

where C is a constant which is independent of k . In particular, the positive measures $\Omega_{\Phi(k)}^{n+1}$ tend weakly to 0.

Proof. Recall that $\dot{\phi}(t; k)$ has been evaluated and is given by (2.16). In particular, it is independent of the choice of metric on L^k . We now apply (2.5) to the path of Fubini-Study metrics $\phi(t; k)$ defined by (1.9) and obtain

$$\int_{X \times A} \Omega_{\Phi(k)}^{n+1} = \frac{2}{k} \int_X \frac{\sum_{j=0}^{N_k} \lambda_j |s_j^{(1)}|^2}{\sum_{j=0}^{N_k} |s_j^{(1)}|^2} \omega_{\phi(1)}^n - \frac{2}{k} \int_X \frac{\sum_{j=0}^{N_k} \lambda_j |s_j^{(0)}|^2}{\sum_{j=0}^{N_k} |s_j^{(0)}|^2} \omega_{\phi(0)}^n. \tag{2.19}$$

Here $\Phi(k)$ is defined by (2.1) with $\phi(t)$ replaced by $\phi(t; k)$. Since $h_1(k) = h_{\hat{g}(1)}$ the formula (1.4) implies

$$k^{-n} \sum_{j=0}^{N_k} |s_j^{(1)}|_{h_1^k(k)}^2 = \sum_{j=0}^{N_k} |\hat{s}_j^{(1)}|_{h_1^k(k)}^2 = 1$$

and, observing that $\omega_{\phi(1)} = \omega_1(k)$ and $\omega_{\phi(0)} = \omega_0(k)$, we can rewrite (2.19) as

$$\begin{aligned} \int_{X \times A} \Omega_{\Phi(k)}^{n+1} &= \frac{2}{k^{n+1}} \int_X \sum_{j=0}^{N_k} \lambda_j |s_j^{(1)}|_{h_1^k(k)}^2 \omega_1^n(k) \\ &\quad - \frac{2}{k^{n+1}} \int_X \sum_{j=0}^{N_k} \lambda_j |s_j^{(0)}|_{h_0^k(k)}^2 \omega_0^n(k). \end{aligned} \tag{2.20}$$

Now observe that

$$\int_X |s_j^{(1)}|_{h_1^k(k)}^2 \omega_1^n(k) = \int_X |s_j^{(1)}|_{h_1^k}^2 \omega_1^n \cdot \frac{h_1^k(k)}{h_1^k} \cdot \frac{\omega_1^n(k)}{\omega_1^n}.$$

On the other hand, (1.7) implies $\frac{h_1^k(k)}{h_1^k} = 1 + O(\frac{1}{k})$ and $\frac{\omega_1^n(k)}{\omega_1^n} = 1 + O(\frac{1}{k^2})$. Moreover, $\int_X |s_j^{(1)}|_{h_1^k}^2 \omega_1^n = 1$ since the $s_j^{(1)}$ are orthonormal with respect to h_1^k . Thus

$$\int_X |s_j^{(1)}|_{h_1^k(k)}^2 \omega_1^n(k) = 1 + O\left(\frac{1}{k}\right)$$

and hence

$$\begin{aligned} \frac{2}{k^{n+1}} \int_X \sum_{j=0}^{N_k} \lambda_j |s_j^{(1)}|_{h_1^k(k)}^2 \omega_1^n(k) - \frac{2}{k^{n+1}} \sum_{j=0}^{N_k} \lambda_j \\ = O\left(\frac{1}{k^{n+2}}\right) \cdot N_k \cdot \max_{0 \leq j \leq N_k} |\lambda_j|. \end{aligned} \quad (2.21)$$

Now the Riemann-Roch theorem implies $N_k = O(k^n)$ and thus we obtain from Lemma 1 that the right side of (2.21) is of the size $O(\frac{1}{k})$. Theorem 2 follows now from (2.20) and (2.21). \square

3. Generalized solutions of the Monge-Ampère equation

In the previous section, we have seen that the functions $\Phi(k)$ form a uniformly bounded sequence of functions whose Monge-Ampère operators $\Omega_{\Phi(k)}^{n+1}$ are positive measures on \bar{M} which tend to 0. The Chern-Levine-Nirenberg inequality [8] implies that if any subsequence of the $\Phi(k)$'s converges uniformly, then its limit Φ would satisfy the Monge-Ampère equation $\Omega_{\Phi}^{n+1} = 0$ in the generalized sense. A major problem is the fact that the bounds available to us at the present time (c.f. Lemma 1) are not strong enough to guarantee the existence of a uniformly convergent subsequence of the $\Phi(k)$'s. Of course, weakly convergent subsequences can always be found. However, it is well-known that the Monge-Ampère operator $PSH \cap C^0 \ni \Phi \rightarrow \Omega_{\Phi}^{n+1}$ is not lower semi-continuous under weak limits [5, 17].

To circumvent these difficulties, we shall formulate and establish extensions of the classical convergence and uniqueness theorems of Bedford-Taylor for the Monge-Ampère operator for domains in \mathbf{C}^n to the case of Kähler manifolds with boundary. We shall then apply the convergence result (Theorem 3) to show that $\Phi = \lim[\sup \Phi(k)]^*$ is a solution to the Dirichlet problem. Then, using the uniqueness theorem (Theorem 6), we shall show that the Φ produced by this limiting method is in fact equal to the $C^{1,1}$ solution whose existence was established by Chen [6].

3.1. Convergence of approximate solutions

The first of these extensions is the following convergence theorem, where the key hypothesis is the existence of uniform C^0 bounds for some transversal derivative of the $\Phi(k)$'s at the boundary of the manifold \bar{M} .

Let \bar{M} be a compact complex manifold with smooth boundary and $M \subseteq \bar{M}$ be the interior of \bar{M} . Let $\bar{M} = \cup_{\alpha=1}^N U_{\alpha}$ be a covering of \bar{M} by a finite number of coordinate charts U_{α} . Fix a smooth closed $(1, 1)$ -form Ω_0 on M , and let Ψ_{α} be smooth potentials for Ω_0 on U_{α} , that is, $\Omega_0 = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Psi_{\alpha}$

on U_α . We define the class of Ω_0 -plurisubharmonic functions on M to be the class of functions Φ on \bar{M} by

$$PSH(M, \Omega_0) = \{\Phi; \Psi_\alpha + \Phi \text{ is plurisubharmonic on } U_\alpha, 1 \leq \alpha \leq N\}. \tag{3.1}$$

Note that this condition means that $\Psi_\alpha + \Phi$ is upper semi-continuous and satisfies the sub-mean value property. However, it is a stronger condition than the condition $\Omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\Phi \geq 0$ by itself, since it is a pointwise condition, while the condition $\Omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\Phi \geq 0$ depends only on the values of Φ almost everywhere.

Next, let ϕ be a continuous function on $\partial\bar{M}$, and Ω_0 a real, smooth closed $(1, 1)$ form on M . Set $d = \partial + \bar{\partial}$, $d^c = \frac{\sqrt{-1}}{4}(\bar{\partial} - \partial)$, so that $dd^c = \frac{\sqrt{-1}}{2} \partial\bar{\partial}$.

Definition 1 Let $\Phi : \bar{M} \rightarrow \mathbf{R}$ be an upper-semicontinuous function. We say Φ is a solution of the Dirichlet problem with boundary values ϕ if

1. $\Phi \in PSH(M, \Omega_0)$
2. Φ is continuous at p for all $p \in \partial\bar{M}$ and $\Phi|_{\partial\bar{M}} = \phi$.
3. $(\Omega_0 + dd^c \Phi)^m = 0$ on \bar{M} where $m = \dim(M)$.

Remark In the case where $\bar{M} = X \times A$ with (X, ω_0) a compact Kähler manifold and A an annulus (or, more generally, A is a compact Riemann surface with smooth boundary), Chen [6] has proved that the Dirichlet problem always has a solution Φ which is $C^{1,1}$.

We wish to obtain a solution of the Dirichlet problem on \bar{M} from a sequence of approximate solutions. Let Y be a smooth real nowhere vanishing vector field on a neighborhood $U \subseteq \bar{M}$ of $\partial\bar{M}$ which is transversal to $\partial\bar{M}$, in the sense that for $p \in \partial\bar{M}$, the vector $Y(p)$ is not tangent to $\partial\bar{M}$.

Theorem 3 Assume $\Omega_0^m = 0$. Let $\Phi_k \in PSH(M, \Omega_0) \cap C^\infty(\bar{M})$ have the properties:

1. $|\Phi_k|$ is uniformly bounded on \bar{M} .
2. $|Y(\Phi_k)|$ is uniformly bounded on U ;
3. $\Phi_k|_{\partial\bar{M}} \rightarrow \phi$ uniformly;
4. There is a sequence $a_k > \sup_{x \in \partial\bar{M}} |\Phi_k(x) - \phi(x)|$ with the property $a_k \searrow 0$ (i.e. a_k is decreasing and approaching zero) and $\sum_k a_k < \infty$.
5. $\lim_{k \rightarrow \infty} \int_M (\Omega_0 + dd^c \Phi_k)^m = 0$

Let

$$\Phi = \lim_{k \rightarrow \infty} [\sup_{l \geq k} \Phi_l]^*.$$

Then Φ is a solution of the Dirichlet problem with boundary values ϕ .

Remark Assumption 4 implies Assumption 3. On the other hand, after passing to a subsequence, Assumption 3 implies Assumption 4.

Proof. We want to choose a sequence $c_k \searrow 0$ in such a way that $\Phi_k + c_k$ is monotonically decreasing on ∂M . To do this, we just define $c_k = 2 \sum_{j \geq k} a_j$. Then $c_k - c_{k+1} = 2a_k$. Moreover, if $x \in \partial M$, then

$$\Phi_k(x) - \Phi_{k+1}(x) > (\phi - a_k) - (\phi + a_{k+1}) \geq -2a_k = c_{k+1} - c_k. \quad (3.2)$$

Thus, replacing Φ_k by $\Phi_k + c_k$, we may assume that $\Phi_k|_{\partial\Omega} > \Phi_{k+1}|_{\partial\Omega}$.

Let

$$W_k = [\sup_{l \geq k} \Phi_l]^*.$$

Then $W_k \in PSH(M, \Omega_0)$ by Theorem 5.7 of [9]. Moreover, (3.2) implies that $W_k = \Phi_k$ in an open neighborhood of $\partial\bar{M}$. We claim that

$$\int_M (\Omega_0 + dd^c W_k)^m = \int_M (\Omega_0 + dd^c \Phi_k)^m. \quad (3.3)$$

This follows from the following simple lemma:

Lemma 2 *Let $W, \Phi \in PSH(M, \Omega_0)$ with $\Omega_0^m = 0$ and assume that $W = \Phi$ on some neighborhood of $\partial\bar{M}$. Then*

$$\int_M (\Omega_0 + dd^c W)^m = \int_M (\Omega_0 + dd^c \Phi)^m.$$

Proof of Lemma 2. To see this, let K be any compact subset such that $W = \Phi$ on $M \setminus K$, and let $\Psi \in C_0^\infty(M)$ be a function with $\Psi = 1$ on a neighborhood of K . Then, expanding $(\Omega_0 + dd^c W)^m$, we obtain

$$\int_M \Psi(\Omega_0 + dd^c W)^m = \int_M \Psi \Omega_0^m + \int_M \Psi dd^c(\Theta_W) = \int_{M \setminus K} dd^c \Psi \wedge \Theta_W.$$

Here $\Theta_W = \sum_{k=0}^{m-1} \binom{m}{k} \Omega_0^k \wedge W \wedge (dd^c W)^{m-1-k}$. Since $\Theta_W = \Theta_\Phi$ on $M \setminus K$, the lemma is proved.

Now Assumption 5 of the theorem together with (3.3) implies that $(\Omega_0 + dd^c W_k)^m \rightarrow 0$ weakly. On the other hand, since $W_k \rightarrow \Phi$ monotonically, the Bedford-Taylor monotonicity theorem (Theorem 2.1 of [2]; this is stated for domains in \mathbf{C}^n , but generalizes in a straightforward fashion to manifolds) implies that $(\Omega_0 + dd^c W_k)^m \rightarrow (\Omega_0 + dd^c \Phi)^m$ weakly. Thus we have $(\Omega_0 + dd^c \Phi)^m = 0$. To finish the proof of the theorem, we must show that Φ is continuous at the boundary and has the right boundary values.

Let $\epsilon > 0$. Choose k_0 such that $k \geq k_0 \implies \sup_{x \in \partial\bar{M}} |\Phi_k(x) - \phi(x)| < \epsilon$. Extend ϕ to a continuous function on a neighborhood $U \subseteq \bar{M}$ of $\partial\bar{M}$ in such a way that ϕ is constant on the flow lines of Y . Then Assumption 2 implies that if U is sufficiently small, then

$$\sup_{x \in U} |\Phi_k(x) - \phi(x)| < 2\epsilon \quad (3.4)$$

and thus $\sup_{x \in U} |\Phi(x) - \phi(x)| \leq 2\epsilon$. In particular, $\Phi|_{\partial\Omega} = \phi$ and Φ is continuous at all points $p \in \partial\bar{M}$. This proves the theorem. \square

3.2. The domination principle for the Monge-Ampère operator

The proof of the extension of the Bedford-Taylor uniqueness theorem in this section follows closely the original arguments of [1,2], and especially the exposition of Blocki [3].

When we consider generalized solutions of a partial differential equation, a particularly desirable property is their uniqueness. For bounded domains in \mathbf{C}^n , the uniqueness of the generalized solution of the Dirichlet problem for the Monge-Ampère equation in the class $PSH \cap L^\infty$ has been established by Bedford and Taylor [2]. It seems that this uniqueness theorem should extend as well to bounded domains in Kähler manifolds, at least if good smooth approximations of $\Omega_{\Phi(0)}$ -plurisubharmonic functions exist. Although there are now many powerful approximation theorems (see [9, 15] and references therein), we found it more convenient to extend the Bedford-Taylor uniqueness theorem to a situation adapted to the problem at hand, in the spirit of the earlier extension. The key hypothesis which we will exploit is a capacity zero condition.

Recall the following notion of capacity of a set introduced by Bedford and Taylor [2]: If $E \subseteq U$ is a Borel subset of a bounded domain $U \subseteq \mathbf{C}^n$ then

$$c(E, U) = \sup \left\{ \int_E (dd^c v)^n; v \in PSH(U), 0 \leq v \leq 1 \right\}. \tag{3.5}$$

For our purposes, we can adapt this notion to Kähler manifolds as follows: Let $M = \cup_{\alpha=1}^N U_\alpha$ be a finite cover of M by coordinate neighborhoods. Then we say $c(E, M) < \epsilon$ if we can write $E = \cup_\alpha E_\alpha$ with $E_\alpha \subseteq U_\alpha$ a Borel subset and

$$\sum_\alpha c(E_\alpha, U_\alpha) < \epsilon. \tag{3.6}$$

We say that $c(E, M) = 0$ if $c(E, M) < \epsilon$ for every $\epsilon > 0$.

Lemma 3 *There is a constant $C > 0$ with the following property: If $E \subseteq M$ is a Borel subset, $\epsilon > 0$ and $\Phi \in PSH(\Omega_0, M)$, then*

$$\int_E (\Omega_0 + dd^c \Phi)^m \leq C\epsilon(1 + \sup |\Phi|)^m \tag{3.7}$$

if $c(E, M) < \epsilon$. In particular, if $c(E, M) = 0$, then for all functions $\Phi \in PSH(\Omega_0, M)$, we have

$$\int_E (\Omega_0 + dd^c \Phi)^m = 0. \tag{3.8}$$

Proof. Fix a smooth potential Ψ_α on U_α such that $\Omega_0 = dd^c \Psi_\alpha$. Then

$$\begin{aligned} \int_E (\Omega_0 + dd^c \Phi)^m &\leq \sum_\alpha \int_{E_\alpha} (\Omega_0 + dd^c \Phi)^m = \sum_\alpha \int_{E_\alpha} (dd^c(\Psi_\alpha + \Phi))^m \\ &\leq \sum_\alpha (\sup |\Psi_\alpha + \Phi|)^m c(E_\alpha, U_\alpha). \end{aligned} \tag{3.9}$$

Since the Ψ_α are fixed, we have $|\Psi_\alpha + \Phi| \leq C(1 + |\Phi|)$ and the lemma follows.

The following lemma follows immediately from the quasi-continuity theorem of Bedford-Taylor [2]:

Lemma 4 *Let $\Phi \in PSH(M, \Omega_0)$. Then for every $\epsilon > 0$, there is an open set $G \subseteq M$ such that $c(G, M) < \epsilon$ and Φ is continuous on $M \setminus G$.*

Proof. Since $\Psi_\alpha + \Phi$ is a plurisubharmonic function on U_α , the quasi-continuity theorem of Bedford-Taylor ([2], Theorem 3.5) implies that there is an open set $G_\alpha \subseteq U_\alpha$ such that $\Psi_\alpha + \Phi$ is continuous on $U_\alpha \setminus G_\alpha$ and $c(G_\alpha, U_\alpha) < \epsilon$. Let $G = \cup_\alpha G_\alpha$. Then Φ is continuous on $M \setminus G$ and, by definition, $c(G, M) < N\epsilon$.

We also require a notion of “nearly continuous” functions:

Definition 2 *We say that a bounded function $v : \bar{M} \rightarrow \mathbf{R}$ is “nearly continuous” if*

1. *There exists a lower semi-continuous function v_0 on \bar{M} such that $v = v_0^*$;*
2. *$\{v_0 < v\}$ has capacity zero, that is, $c(\{v_0 < v\}, M) = 0$;*
3. *$v = v_0$ on ∂M .*

With this notion, we shall prove the following:

Theorem 4 *Assume that \bar{M} is a complex manifold of dimension m with smooth boundary, let $M = \bar{M} \setminus \partial M$ and let Ω_0 be a real closed smooth $(1, 1)$ form on M satisfying $\Omega_0^m = 0$. Let $u, v \in PSH(M, \Omega_0) \cap L^\infty$ be such that $(u - v)_* \geq 0$ on $\partial \bar{M}$. Assume as well:*

1. *u is continuous;*
2. *There is a decreasing sequence v_k of nearly continuous functions in $PSH(M, \Omega_0)$ such that $v_k \searrow v$;*
3. *For every $\delta > 0$ there is a compact set $K \subseteq M$ such that $v_k|_{M \setminus K} < v|_{M \setminus K} + \delta$ for k sufficiently large.*

Then

$$\int_{u < v} (\Omega_0 + dd^c v)^m \leq \int_{u < v} (\Omega_0 + dd^c u)^m. \tag{3.10}$$

Proof. We divide the proof into several steps.

Step 1. If we replace u by $u + \delta$, then we have $\{u + \delta < v\} \uparrow \{u < v\}$ as $\delta \downarrow 0$. Since for any positive measure μ we have $\mu(E_j) \rightarrow \mu(\cup E_j)$ whenever E_j is an increasing family of measurable sets, we may replace u by $u + \delta$. Thus we may assume that

$$(u - v)_* \geq \delta \text{ and that } M' = \{u < v\} \text{ is relatively compact in } M. \quad (3.11)$$

Step 2. We prove the theorem under the assumption that v is continuous (in which case the Hypotheses 2 and 3 are automatic, since we can take $v_k = v$ for all k).

For $\epsilon > 0$ let $u_\epsilon = \max(u + \epsilon, v)$. Then $u_\epsilon = u + \epsilon$ on a neighborhood of $\partial M'$. Since $dd^c(u + \epsilon) = dd^c u$, we can invoke Lemma 2 and conclude that

$$\int_{M'} (\Omega_0 + dd^c u_\epsilon)^m = \int_{M'} (\Omega_0 + dd^c u)^m. \quad (3.12)$$

On the other hand, $u_\epsilon \downarrow v$ on M' and, by the Bedford-Taylor monotonicity theorem,

$$(\Omega_0 + dd^c u_\epsilon)^m \rightarrow (\Omega_0 + dd^c v)^m \text{ on } M' \text{ (weak convergence of measure).}$$

Since $\{u < v\}$ is open, we obtain

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \liminf_{\epsilon \rightarrow 0} \int_{\{u < v\}} (\Omega_0 + dd^c u_\epsilon)^m.$$

This completes Step 2.

Step 3. Now we treat the case where v itself is nearly continuous (in which case the Hypotheses 2 and 3 are again automatic, since we can take $v_k = v$ for all k). This step will be parallel to the argument in Step 2:

For $\epsilon > 0$ let $u_\epsilon = \max(u + \epsilon, v)$. Then $u_\epsilon \searrow v$ on the open set $\{u < v_0\}$ (this set is open since v_0 is lower semi-continuous). In particular, the Bedford-Taylor monotonicity theorem implies $(\Omega_0 + dd^c u_\epsilon)^m \rightarrow (\Omega_0 + dd^c v)^m$ weakly on $\{u < v_0\}$ (as measures or as currents - the two notions of weak convergence are equivalent). Thus we have

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c v)^m &= \int_{\{u < v_0\}} (\Omega_0 + dd^c v)^m \\ &\leq \liminf_{\epsilon \rightarrow 0} \int_{\{u < v_0\}} (\Omega_0 + dd^c u_\epsilon)^m \\ &= \liminf_{\epsilon \rightarrow 0} \int_{\{u < v\}} (\Omega_0 + dd^c u_\epsilon)^m. \end{aligned}$$

The first equality follows from the assumption that $\{v_0 < v\}$ has capacity zero and the inequality from the fact that $(\Omega_0 + dd^c u_\epsilon)^m \rightarrow (\Omega_0 + dd^c v)^m$ weakly. Here we are making strong use of the fact that $\{u < v_0\}$ is open.

Next we claim that for all $\epsilon > 0$:

$$\int_{\{u < v\}} (\Omega_0 + dd^c u_\epsilon)^m = \int_{\{u < v\}} (\Omega_0 + dd^c u)^m. \quad (3.13)$$

To see this, let A be an open set containing $\{u < v\}$, which is relatively compact in M . Then, $p \in \partial A$ implies that $p \notin A$, and we have $u(p) \geq v(p)$, so $u_\epsilon(p) = u(p) + \epsilon > v(p)$. Since the set $\{u + \epsilon > v\}$ is open (u is continuous and v is upper semi-continuous), we see that $u + \epsilon = u_\epsilon$ in a neighborhood of ∂A . Thus Lemma 2 implies

$$\int_A (\Omega_0 + dd^c u_\epsilon)^m = \int_A (\Omega_0 + dd^c (u + \epsilon))^m = \int_A (\Omega_0 + dd^c u)^m.$$

Since $(\Omega_0 + dd^c u_\epsilon)^m$ and $(\Omega_0 + dd^c u)^m$ are positive Borel measures which are finite on compact subsets of M , they are both regular. In particular

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c u_\epsilon)^m &= \inf_A \int_A (\Omega_0 + dd^c u_\epsilon)^m \\ \int_{\{u < v\}} (\Omega_0 + dd^c u)^m &= \inf_A \int_A (\Omega_0 + dd^c u)^m \end{aligned} \quad (3.14)$$

where the inf is taken over all open sets A which contain $\{u < v\}$. This proves (3.13).

Step 4. We treat the general case. From Step 1 we can assume that $(u - v)_* \geq \delta > 0$. Thus $u > v_k$ on a neighborhood of ∂M for k sufficiently large. By Lemma 4, there is an open set $G \subseteq M$ such that $c(G, M) < \epsilon$ (the capacity of G in M) and such that v is continuous on $F = M \setminus G$. Choose ϕ which is continuous M such that $\phi = v$ on F . Then $\{u < v\} \subseteq \{u < \phi\} \cup G$ implies

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c v)^m &\leq \int_{\{u < \phi\}} (\Omega_0 + dd^c v)^m + \int_G (\Omega_0 + dd^c v)^m \\ &\leq \int_{\{u < \phi\}} (\Omega_0 + dd^c v)^m + C\epsilon \end{aligned} \quad (3.15)$$

where the last inequality follows from Lemma 3, the fact that $c(G, M) < \epsilon$ and that the function v is bounded. Here the constant C depends only on the sup norm of v .

Since $\{u < \phi\}$ is open and $v_k \searrow v$, the Bedford-Taylor monotonicity theorem implies

$$\int_{\{u < \phi\}} (\Omega_0 + dd^c v)^m \leq \liminf_{k \rightarrow \infty} \int_{\{u < \phi\}} (\Omega_0 + dd^c v_k)^m.$$

Now

$$\{u < \phi\} \subseteq \{u < v\} \cup G \subseteq \{u < v_k\} \cup G.$$

Thus

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \liminf_{k \rightarrow \infty} \int_{\{u < v_k\}} (\Omega_0 + dd^c v_k)^m + 2C\epsilon,$$

since our assumptions imply that the functions v_k are uniformly bounded. Next, the Assumption 3 implies that the sets $\{u < v_k\}$ are all contained in a relatively compact subset of M . Using Step 3,

$$\int_{\{u < v_k\}} (\Omega_0 + dd^c v_k)^m \leq \int_{\{u < v_k\}} (\Omega_0 + dd^c u)^m$$

and since $v_k \searrow v$, $\cap_k \{u < v_k\} = \{u \leq v\}$, we can conclude that

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u \leq v\}} (\Omega_0 + dd^c u)^m + 2C\epsilon.$$

Since ϵ is arbitrary:

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u \leq v\}} (\Omega_0 + dd^c u)^m.$$

Applying this last inequality to $u + \eta$ and v , for some $\eta > 0$:

$$\int_{\{u + \eta < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u + \eta \leq v\}} (\Omega_0 + dd^c u)^m.$$

Finally, taking the limit as $\eta \rightarrow 0$ and noting that $\cup_{\eta > 0} \{u + \eta < v\} = \cup_{\eta > 0} \{u + \eta \leq v\} = \{u < v\}$, we obtain (3.16).

Next we prove another version of the domination theorem, but this time with the roles of u, v reversed:

Theorem 5 *Assume that \bar{M} is a complex manifold of dimension m with smooth boundary, let $M = \bar{M} \setminus \partial M$ and let Ω_0 be a real closed smooth $(1, 1)$ form on M with the property: $\Omega_0^m = 0$. Let $u, v \in \text{PSH}(\bar{M}, \Omega_0) \cap L^\infty$ satisfy $(u - v)_* \geq 0$ on $\partial \bar{M}$. Assume as well:*

1. v is continuous;
2. There is a decreasing sequence u_k of nearly continuous PSH functions on M such that $u_k \searrow u$.

Then

$$\int_{u < v} (\Omega_0 + dd^c v)^m \leq \int_{u < v} (\Omega_0 + dd^c u)^m. \quad (3.16)$$

Proof. The proof is parallel to that of Theorem 4, although there are some important differences. We again divide it into several steps.

Step 1. As before, we may assume that

$$(u - v)_* \geq \delta \text{ and that } M' = \{u < v\} \text{ is relatively compact in } M.$$

Moreover, we have

$$\{u_0 < v\} \text{ is relatively compact in } M.$$

To see this, observe first that $u \geq v + \delta/2$ in an open neighborhood of ∂M . Also, $\{u_0 - u > -\delta/4\}$ is open since u_0 is lower semi-continuous and u is upper semi-continuous. Since $u_0 = u$ on ∂M , the set $\{u_0 - u > -\delta/4\}$ is an open neighborhood of ∂M . Thus $u_0 > v + \delta/4$ in an open neighborhood of ∂M .

Step 2. Now we treat the case where u itself is nearly continuous (in which case Hypotheses 2 and 3 are automatic, since we can take $u_k = u$ for all k).

For $\epsilon > 0$ let $u_\epsilon = \max(u + \epsilon, v)$. Then $u_\epsilon \searrow v$ on the open set $\{u < v\}$ (this set is open since u is upper semi-continuous). As before, the Bedford-Taylor monotonicity theorem implies $(\Omega_0 + dd^c u_\epsilon)^m \rightarrow (\Omega_0 + dd^c v)^m$ weakly and, making use of the fact that $\{u < v\}$ is open, we get

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c v)^m &\leq \liminf_{\epsilon \rightarrow 0} \int_{\{u < v\}} (\Omega_0 + dd^c u_\epsilon)^m \\ &= \liminf_{\epsilon \rightarrow 0} \int_{\{u_0 < v\}} (\Omega_0 + dd^c u_\epsilon)^m, \end{aligned} \tag{3.17}$$

where u_0 is the function which appears in the definition of the near continuity of u . Next we claim that for all $\epsilon > 0$:

$$\int_{\{u_0 < v\}} (\Omega_0 + dd^c u_\epsilon)^m = \int_{\{u_0 < v\}} (\Omega_0 + dd^c u)^m. \tag{3.18}$$

The proof is similar to that of (3.13), using A an open set containing $\{u_0 < v\}$, $u_0(p) \geq v(p)$ for $p \in \partial A$ so that $u_0(p) + \epsilon > v(p)$. We use now the continuity of v and the lower semi-continuity of u_0 to deduce that the set $\{u_0 + \epsilon > v\}$ is open, and $u + \epsilon = u_\epsilon$ in a neighborhood of ∂A . As before, Lemma 2 implies

$$\int_A (\Omega_0 + dd^c u_\epsilon)^m = \int_A (\Omega_0 + dd^c (u + \epsilon))^m = \int_A (\Omega_0 + dd^c u)^m, \tag{3.19}$$

and, using the fact that $(\Omega_0 + dd^c u_\epsilon)^m$ and $(\Omega_0 + dd^c u)^m$ are positive Borel measures which are finite on compact subsets of M ,

$$\begin{aligned} \int_{\{u_0 < v\}} (\Omega_0 + dd^c u_\epsilon)^m &= \inf_A \int_A (\Omega_0 + dd^c u_\epsilon)^m \\ \int_{\{u_0 < v\}} (\Omega_0 + dd^c u)^m &= \inf_A \int_A (\Omega_0 + dd^c u)^m \end{aligned} \tag{3.20}$$

where the inf is taken over all open sets A which contain $\{u_0 < v\}$. This proves (3.18). Since u and u_0 differ only on a set of capacity 0, we obtain the desired inequality.

Step 4. We treat the general case: From Step 1 we can assume that $(u - v)_* \geq \delta > 0$. Thus $u_j > v$ on a neighborhood of ∂M for j sufficiently large. Choose an open set $G \subseteq M$ such that $c(G, M) < \epsilon$ (the capacity of G in M) and such that u is continuous on $F = M \setminus G$. Choose ϕ which is continuous M such that $\phi = u$ on F . Now

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c v)^m &= \lim_{j \rightarrow \infty} \int_{\{u_j < v\}} (\Omega_0 + dd^c v)^m \\ &\leq \lim_{j \rightarrow \infty} \int_{\{u_j < v\}} (\Omega_0 + dd^c u_j)^m \\ &\leq \liminf_{j \rightarrow \infty} \int_{\{u < v\}} (\Omega_0 + dd^c u_j)^m \end{aligned} \tag{3.21}$$

where we have made use of Step 3 and the fact that $u_j > v$ on ∂M to prove the first inequality. Now we have $\{u < v\} \subseteq \{\phi < v\} \cup G \subseteq (K \cap \{\phi \leq v\}) \cup G$ where $K \subseteq M$ is a compact set such that $\{u < v\} \subseteq K$ so

$$\begin{aligned} \int_{\{u < v\}} (\Omega_0 + dd^c u_j)^m &\leq \int_{\{\phi \leq v\} \cap K} (\Omega_0 + dd^c u_j)^m + \int_G (\Omega_0 + dd^c u_j)^m \\ &\leq \int_{\{\phi \leq v\} \cap K} (\Omega_0 + dd^c u_j)^m + C\epsilon \end{aligned} \tag{3.22}$$

where the last inequality follows from the fact that $c(G, M) < \epsilon$ and that the function v is bounded. Here the constant C depends only on the sup norm of v .

Since $\{\phi \leq v\} \cap K$ is compact, the Bedford-Taylor monotonicity theorem implies

$$\limsup_{k \rightarrow \infty} \int_{\{\phi \leq v\} \cap K} (\Omega_0 + dd^c u_j)^m \leq \int_{\{\phi \leq v\} \cap K} (\Omega_0 + dd^c u)^m.$$

Now

$$\{\phi \leq v\} \cap K \subseteq \{\phi \leq v\} \subseteq \{u \leq v\} \cup G.$$

Thus

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u \leq v\}} (\Omega_0 + dd^c u)^m + 2C\epsilon.$$

Since ϵ is arbitrary:

$$\int_{\{u < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u \leq v\}} (\Omega_0 + dd^c u)^m.$$

Applying this last inequality to $u + \eta$ and v , for some $\eta > 0$:

$$\int_{\{u + \eta < v\}} (\Omega_0 + dd^c v)^m \leq \int_{\{u + \eta \leq v\}} (\Omega_0 + dd^c u)^m.$$

Finally, taking the limit as $\eta \rightarrow 0$, we obtain (3.16).

We can now state and prove the uniqueness theorem which we need for the proof of Theorem 1:

Theorem 6 *Let u, v be as in Theorem 4, that is,*

1. u is continuous;
2. There is a decreasing sequence v_k of nearly continuous PSH functions on M such that $v_k \searrow v$;
3. For every $\delta > 0$ there is a compact set $K \subseteq M$ such that $v_k|_{M \setminus K} < v|_{M \setminus K} + \delta$ for k sufficiently large.

Assume that there exists some $\psi \in C^\infty(\bar{M})$ such that $\Omega_0 + dd^c \psi > 0$. Assume that $(u - v)_* = (u - v)^* = 0$ on ∂M and that $(\Omega_0 + dd^c u)^m = (\Omega_0 + dd^c v)^m = 0$. Then $u = v$.

Proof. We may assume, after replacing u and v by $u + C$ and $v + C$ for some constant C , that u and v are positive. We wish to show $u = v$. Assume not: Let $\psi \in C^\infty(\bar{M})$ be such that $\Omega_0 + dd^c \psi > 0$. Replacing ψ by $\psi - C$, we may assume that $\psi < 0$ on \bar{M} .

Case 1. $\{u < v\} \neq \emptyset$. This implies $\{u < (1 - \epsilon)v + \epsilon\psi\} \neq \emptyset$ for some $\epsilon > 0$. Let $p \in \{u < (1 - \epsilon)v + \epsilon\psi\}$ and let D be a disk in some coordinate neighborhood of p . Then $D \cap \{u < (1 - \epsilon)v + \epsilon\psi\}$ has non-zero Lebesgue measure (in general, if u, v are psh functions such that $u = v$ almost everywhere in a disk D , then $u = v$ everywhere in D ; this follows from local regularization). Now we have, using Theorem 4

$$\begin{aligned} 0 &= \int_{\{u < (1 - \epsilon)v + \epsilon\psi\}} (\Omega_0 + dd^c u)^m \\ &\geq \int_{\{u < (1 - \epsilon)v + \epsilon\psi\}} (\Omega_0 + dd^c [(1 - \epsilon)v + \epsilon\psi])^m \\ &\geq \epsilon^m \int_{\{u < (1 - \epsilon)v + \epsilon\psi\}} (\Omega_0 + dd^c \psi)^m > 0. \end{aligned} \tag{3.23}$$

Case 2. $\{v < u\} \neq \emptyset$. This is treated exactly in the same way as in Case 1 except that we use Theorem 5 instead of Theorem 4. \square

4. Proof of Theorem 1

We can give now the proof of Theorem 1.

First, we apply Theorem 3 to construct a generalized solution of the Dirichlet problem for the Monge-Ampère equation

$$\Omega_{\Phi}^{n+1} = 0 \text{ on } M = X \times A, \quad \Phi|_{\partial M} = \phi, \tag{4.1}$$

where $\phi : \partial M \rightarrow \mathbf{R}$ is defined by $\phi|_{|w|=1} = 0$ and $\phi|_{|w|=e} = \log \frac{h_0}{h_1}$. Define $\Phi(k)(z, w) = \phi(t; k)$, where $t = \log |w|$ and $\phi(t; k)$ is defined as in (1.9). Let $Y = \partial_t$. Then $|\Phi(k)| \leq C$ and $|Y(\Phi(k))| \leq C$ by Lemma 1. We also have $\|\Phi(k)|_{\partial \bar{M}} - \phi\|_{L^\infty} \leq C \frac{1}{k^2}$ by (1.7), so that $\Phi(k)|_{\partial \bar{M}} \rightarrow \phi$ uniformly. Furthermore, Theorem 2 implies that $\int_M \Omega_{\Phi(k)}^{n+1} \rightarrow 0$ as $k \rightarrow \infty$. Thus, since $\sum_{k=1}^\infty \frac{1}{k^2} < \infty$, we can apply Theorem 3, and conclude that

$$\Phi = \lim_{k \rightarrow \infty} [\sup_{l \geq k} \Phi(l)]^* \tag{4.2}$$

is a generalized solution of the desired Dirichlet problem (4.1).

Consider next the $C^{1,1}$ geodesic ϕ_t joining ϕ_0 to ϕ_1 in the space \mathcal{H} of Kähler potentials: This means that if $\tilde{\Phi}(z, w) = \phi_t(z)$ with $t = \log |w|$, then $\tilde{\Phi}$ is a $C^{1,1}$ solution to the Dirichlet problem whose boundary values are the same as those for Φ . We shall show that $\Phi = \tilde{\Phi}$. To do this, we would like to apply Theorem 6 with

$$u = \tilde{\Phi}, \quad (v_k)_0 = \sup_{l \geq k} \Phi(l), \quad v_k = (v_k)_0^*, \quad v = \Phi = \lim_{k \rightarrow \infty} v_k. \tag{4.3}$$

First, we show that v_k is nearly continuous. Since $\Phi(l)$ is smooth, $(v_k)_0$ is lower semi-continuous. Moreover, $\{(v_k)_0 < v_k\}$ has capacity 0, by Proposition 5.1 of [2]. It remains to show that $v_k = (v_k)_0$ on $\partial \bar{M}$. The equation (1.7) implies $|D\Phi(l)(z, w)| \leq C$, for some constant C independent of l , if $(z, w) \in \partial \bar{M}$ and D is any derivative tangent to $\partial \bar{M}$. Thus, if $\delta > 0$, there exists $\epsilon > 0$ such that if $(z_0, w_0) \in \partial \bar{M}$ then

$$\Phi(l)(z_1, w_0) + \delta > \Phi(l)(z_0, w_0) > \Phi(l)(z_1, w_0) - \delta, \tag{4.4}$$

for all l if $|z_1 - z_0| < \epsilon$. Also, $|Y(\Phi(l))| \leq C$ implies that

$$\Phi(l)(z_1, w_1) + \delta > \Phi(l)(z_1, w_0) > \Phi(l)(z_1, w_1) - \delta \text{ if } |w_0 - w_1| < \epsilon. \tag{4.5}$$

We have then $\Phi(l)(z_1, w_1) + 2\delta > \Phi(l)(z_0, w_0) > \Phi(l)(z_1, w_1) - 2\delta$, which implies that

$$(v_k)_0(z_1, w_1) + 2\delta > (v_k)_0(z_0, w_0) > (v_k)_0(z_1, w_1) - 2\delta, \tag{4.6}$$

and hence

$$(v_k)_0(z_1, w_1) + 2\delta > (v_k)_0(z_0, w_0) > v_k(z_1, w_1) - 2\delta, \tag{4.7}$$

if $|z_1 - z_0| < \epsilon, |w_1 - w_0| < \epsilon$. In particular, $(v_k)_0(z_0, w_0) > v_k(z_0, w_0) - 2\delta$ for all δ so $(v_k)_0 = v_k$ on ∂M so v_k is indeed nearly continuous.

Thus the first two assumptions of Theorem 6 are satisfied.

In the next step, we will need the bound

$$v(z_1, w_1) + 2\delta > v(z_0, w_0) > v(z_1, w_1) - 2\delta, \tag{4.8}$$

which follows by taking the limit of (4.7) as $k \rightarrow \infty$.

Next, we verify Assumption 3: Since $v_k = (v_k)_0$ on ∂M we see that v_k is both upper and lower semi-continuous on ∂M and thus v_k is continuous on ∂M . Moreover, by Theorem 3, $v = \phi$ on ∂M so v is also continuous on ∂M . Since $v_k \searrow v$ we see, by Dini’s theorem, that $v_k \searrow v$ uniformly on ∂M . Thus, for every $\delta > 0$ (4.7) and (4.8) imply that for $k \gg 0$

$$v_k(z_1, w_1) - 2\delta < v_k(z_0, w_0) < v(z_0, w_0) + \delta < v(z_1, w_1) + 3\delta$$

for all $|z_1 - z_0| < \epsilon$ and $|w_1 - w_0| < \epsilon$.

Now we let $\psi(w) = |w|^2$. Then $\Omega_0 + dd^c \psi > 0$ since it is simply the product metric on $X \times A$ for the given metric ω_0 on X and the Euclidean metric $dd^c(|w|^2)$ on A .

All the conditions of Theorem 6 are satisfied. We can thus conclude that $\Phi = \check{\Phi}$.

Finally, the uniform convergence of the functions $[\sup_{k \geq l} \phi(t, k)]^*$ follows from their upper semi-continuity and the compactness of X . This is essentially Dini’s theorem, and can be proven as follows. Assume that u_n is a sequence of upper semi-continuous functions, decreasing to a continuous limit u . For each $\epsilon > 0$, the sets $\{x \in X; u_n(x) - u(x) < \epsilon\}$ form an open covering of X . Since X is compact, it admits a finite sub-cover, and since the sets are increasing as n increases, we must have $X = \bigcap_{n \geq N_\epsilon} \{x \in X; u_n(x) - u(x) < \epsilon\}$ for some N_ϵ . \square

5. Remarks

We conclude with a few remarks.

- In [11], Donaldson asks when two Kähler metrics can be connected by a smooth geodesic. One way to approach this problem is to establish a priori bounds on the derivatives of $\phi(t; k)$. This was done in Lemma 1 for ϕ and $\dot{\phi}$. Let us now consider $\ddot{\phi}$:

$$\ddot{\phi}(0) = \frac{1}{k} \sum (\lambda_i - \lambda_j)^2 |s_i|_{h_0^k(k)}^2 |s_j|_{h_0^k(k)}^2. \tag{5.1}$$

Note that Lemma 1 implies $|\ddot{\phi}(0)| \leq Ck$, but this is not strong enough, since we need a bound which is independent of k .

Define a random variable Z whose probability distribution is given by

$$P(Z = \lambda_j) = |s_j(z)|_{h_0^k}^2.$$

This is indeed a distribution since the total probability is $\sum_j |s_j(z)|_{h_0^k}^2 = 1$. Moreover, $\sum (\lambda_i - \lambda_j)^2 |s_i|_{h_0^k}^2 |s_j|_{h_0^k}^2$ is just the variance of Z . Thus we need to prove that the variance of Z is bounded by k . In the simplest case where $X = \mathbf{P}^1$ and the line bundle $L = O(1)$ and the metric h is the Fubini-Study metric, then an easy computation shows that Z is just the binomial distribution with k trials, where the probability p of flipping heads is a function of $z \in \mathbf{P}^1$. As is well known, the variance of the binomial distribution is kp , which is the bound we need.

In the case where ω is the Fubini-Study metric on \mathbf{P}^1 , the eigenvalues of the change of basis matrix are just $0, 1, \dots, k$. More generally, we can show that if ϕ is a radially symmetric Kähler potential on \mathbf{P}^1 , and if $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_k$ are the eigenvalues of the change of basis matrix, then $|\lambda_j - \lambda_{j+1}| \leq C$ for some constant C , independent of k . From this one can show without difficulty that $\ddot{\phi}(0)$ is uniformly bounded.

- The function $\lim_{l \rightarrow \infty} [\sup_{k \geq l} \phi(t; k)]^*(x)$ is equal almost everywhere to the $\limsup \phi(t; k)$. The convergence can also be guaranteed to take place in a Sobolev norm of positive order. Indeed, quite generally, the L^2 -norm of $\partial\phi$ can be bounded by $\|\phi\|_{C^0}$ if ϕ is $PSH(X, \omega_0)$ -plurisubharmonic. Indeed,

$$\begin{aligned} \|\partial\phi\|_{\omega_0}^2 &= \int_X \partial\phi \wedge \bar{\partial}\phi \wedge \omega_0^{n-1} = \int_X \phi \partial\bar{\partial}\phi \wedge \omega_0^{n-1} \\ &= \int_X \phi \omega_\phi \wedge \omega_0^{n-1} - \int_X \phi \omega_0^n \end{aligned} \tag{5.2}$$

and the right hand side can be bounded in turn by

$$\begin{aligned} \left| \int_X \phi \omega_\phi \wedge \omega_0^{n-1} - \int_X \phi \omega_0^n \right| &\leq \|\phi\|_{C^0} \left(\int_X \omega_\phi \wedge \omega_0^{n-1} + \int_X \omega_0^n \right) \\ &= 2\|\phi\|_{C^0} \int_X \omega_0^n. \end{aligned} \tag{5.3}$$

(In fact, the same argument gives the following useful inequality

$$J_{\omega_0}(\phi) \leq 2n \|\phi\|_{C^0}, \tag{5.4}$$

where $J_{\omega_0}(\phi) = V^{-1} \sqrt{-1} \sum_{i=0}^{n-1} \frac{n-i}{n+1} \int_X \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^i \wedge \omega_0^{n-i-1}$, $V = \int_X \omega_0^n$, is the familiar Aubin-Yau functional.) Returning to the problem at hand, we deduce that the $H_{(1)}(X \times [0, 1])$ Sobolev norms of the functions $\phi(t; k)$ are uniformly bounded. The same is true for the $H_{(1)}(X)$ Sobolev norms of $\phi(t; k)$ for each $t \in [0, 1]$.

- The functions $\dot{\phi}(t; k)$ also satisfy an interesting Harnack inequality of Li-Yau type. Let $0 \leq \tau < T \leq 1$ and let $\xi, X \in M$. Then we claim

$$\dot{\phi}(\xi, \tau) \leq \dot{\phi}(X, T) + \frac{1}{8} \Delta(\xi, \tau, X, T) \tag{5.5}$$

where

$$\Delta(\xi, \tau, X, T) = \inf_{\gamma} \int_{\tau}^T \left(\frac{ds}{dt} \right)^2 dt \tag{5.6}$$

where $\frac{ds}{dt}$ is the velocity in space at time t and the infimum is taken over all paths from (ξ, τ) to (X, T) parametrized by $\tau \leq t \leq T$. To see this, let $L = 2\dot{\phi}$. Then $\ddot{\phi} - |\partial\dot{\phi}|_{\omega_{\phi}}^2 = |\pi_N V|^2 \geq 0$, where $\pi_N V$ is the normal component of the holomorphic vector field V on \mathbf{CP}^{N_k} generated by σ^t [23], and so

$$\frac{\partial L}{\partial t} \geq |DL|_{\phi}^2. \tag{5.7}$$

As in [18], we can now choose a path $(t, s(t))$ joining (ξ, τ) to (X, T) where $\tau < T$ and $\xi, X \in M$. Then

$$\begin{aligned} L(X, T) - L(\xi, \tau) &= \int_{\tau}^T \frac{dL}{dt} dt = \int_{\tau}^T \left\{ \frac{\partial L}{\partial t} + \frac{\partial L}{\partial s} \cdot \frac{ds}{dt} \right\} dt \\ &\geq \int_{\tau}^T \left\{ |DL|^2 + \frac{\partial L}{\partial s} \cdot \frac{ds}{dt} \right\} dt. \end{aligned} \tag{5.8}$$

Completing the square, we obtain

$$L(X, T) - L(\xi, \tau) \geq -\frac{1}{4} \int \left(\frac{ds}{dt} \right)^2 dt. \tag{5.9}$$

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